LA-UR-12-21217

Approved for public release; distribution is unlimited.

Title: On the Group-Theoretic Structure of Lifted Filter Banks

Second revision with final proof corrections
Dated 6 Aug. 2013

Author(s): Christopher M. Brislawn

Preprint: http://viz.lanl.gov/paper.html

Intended for: Excursions in Harmonic Analysis, vol. 2
ed. Travis Andrews, Radu V. Balan, John J. Benedetto, Wojciech Czaja and Kasso Okoudjou
Birkhäuser Series in Applied and Numerical Harmonic Analysis (2013), pp. 113–135

Los Alamos
NATIONAL LABORATORY
EST. 1943

On the Group-Theoretic Structure of Lifted Filter Banks

Christopher M. Brislawn
(final corrections, 8/6/13)

Abstract The polyphase-with-advance matrix representations of whole-sample symmetric (WS) unimodular filter banks form a multiplicative matrix Laurent polynomial group. Elements of this group can always be factored into lifting matrices with half-sample symmetric (HS) off-diagonal lifting filters; such linear phase lifting factorizations are specified in the ISO/IEC JPEG 2000 image coding standard. Half-sample symmetric unimodular filter banks do not form a group, but such filter banks can be *partially* factored into a cascade of whole-sample *antisymmetric* (WA) lifting matrices starting from a concentric, equal-length HS base filter bank. An algebraic framework called a *group lifting structure* has been introduced to formalize the group-theoretic aspects of matrix lifting factorizations. Despite their pronounced differences, it has been shown that the group lifting structures for both the WS and HS classes satisfy a *polyphase order-increasing property* that implies uniqueness ("modulo rescaling") of irreducible group lifting factorizations in both group lifting structures. These unique factorization results can in turn be used to characterize the group-theoretic structure of the groups generated by the WS and HS group lifting structures.

Key words: Lifting, Filter bank, Linear phase filter, Group theory, Group lifting structure, JPEG 2000, Wavelet, Polyphase matrix, Unique factorization, Matrix polynomial

1 Introduction

Lifting [22, 23, 9] is a general technique for factoring the polyphase matrix representation of a perfect reconstruction multirate filter bank into elementary matrices over the Laurent polynomials. As one might expect of a technique as univer-

---

Christopher M. Brislawn
Los Alamos National Laboratory, Los Alamos, NM, 87545, e-mail: brislawn@lanl.gov

sal as elementary matrix factorization, lifting has proven extremely useful for both theoretical investigations and practical applications. For instance, lifting forms the basis for specifying discrete wavelet transforms in the ISO/IEC JPEG 2000 standards [12, 13].

In addition to providing a completely general mathematical framework for standardizing discrete wavelet transforms, lifting also provides a cascade structure for *reversible* filter banks—nonlinear implementations of linear filter banks that furnish bit-perfect invertibility in fixed-precision arithmetic [5, 19, 26, 6]. Reversibility allows digital communications systems to realize the efficiency and scalability of subband coding while also providing the option of lossless transmission, a key feature that made lifting a particularly attractive choice for the JPEG 2000 standard.

The author became acquainted with lifting while serving on the JPEG 2000 standard, and he was struck by the group-theoretic flavor of the subject. After completing his standards committee work, he began studying the lifting structure of two-channel linear phase FIR filter banks in depth, leading to the publications outlined in the present paper. In spite of its universality, lifting is not particularly well-suited for analyzing *paraunitary* filter banks because, as discussed in [1, Section IV], lifting matrices are never paraunitary. This means lifting factorization takes place *outside* of the paraunitary group, whereas we shall show that lifting factorization can be defined to take place entirely *within* the group of whole-sample symmetric (WS, or odd-length linear phase) filter banks by decomposing WS filter banks into linear phase lifting steps. This allows us to prove both existence and (rather surprisingly) uniqueness of "irreducible" WS group lifting factorizations. One consequence of this unique factorization theory is that we can characterize the group-theoretic structure of the unimodular WS filter bank group up to isomorphism using standard group-theoretic constructs.

Besides WS filter banks, there is also a class of half-sample symmetric (HS, or even-length linear phase) filter banks. The differences between the group-theoretic structure of WS and HS filter banks are striking. For instance, HS filter banks do *not* form a matrix group, but linear phase "partial" lifting factorizations partition the class of unimodular HS filter banks into *cosets* of a particular matrix group generated by whole-sample *antisymmetric* lifting filters. The complete group-theoretic classification of unimodular HS filter banks is still incomplete as of this writing but comprises an extremely active area of research by the author.

The present paper is an expository overview of recent research [4, 1, 2, 3]. It is targeted at a mathematical audience that has at least a passing familiarity with elementary group theory and with the connections between wavelet transforms and multirate filter banks.

1.1 Perfect Reconstruction Filter Banks

This paper studies two-channel multirate digital filter banks of the form shown in Figure 1 [7, 8, 24, 25, 21, 15]. We only consider systems in which both the

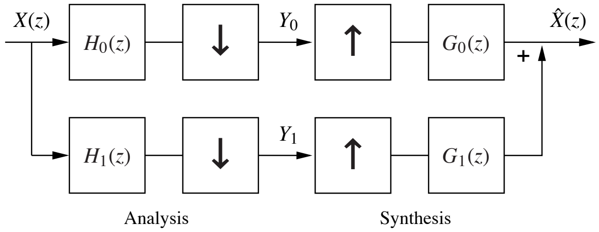

Fig. 1 Two-channel perfect reconstruction multirate filter bank.

analysis filters $\{H_0(z),\, H_1(z)\}$ and the synthesis filters $\{G_0(z),\, G_1(z)\}$ are linear translation-invariant (or time-invariant) finite impulse response (FIR) filters. A system like that in Figure 1 is called a *perfect reconstruction multirate filter bank* (frequently abbreviated to just "filter bank" in this paper) if it is a linear translation-invariant system with a transfer function satisfying

$$\frac{\hat{X}(z)}{X(z)} = az^{-d} \tag{1}$$

for some integer $d \in \mathbb{Z}$ and some constant $a \neq 0$.

FIR filters are written in the transform domain as Laurent polynomials,

$$F(z) \equiv \sum_{n=a}^{b} f(n)\, z^{-n} \in \mathbb{C}\left[z, z^{-1}\right],$$

with impulse response $f(n)$. The *support interval* of an FIR filter, denoted

$$\text{supp\_int}(F) \equiv \text{supp\_int}(f) \equiv [a,b] \subset \mathbb{Z}, \tag{2}$$

is the smallest closed interval of integers containing the support of the filter's impulse response or, equivalently, the largest closed interval for which $f(a) \neq 0$ and $f(b) \neq 0$. If supp_int$(f) = [a,b]$ then the *order* of the filter is

$$\text{order}(F) \equiv b - a. \tag{3}$$

1.2 The Polyphase-with-Advance Representation

It is more efficient to compute the decimated output of a filter bank like the one in Figure 1 by splitting the signal into even- and odd-indexed subsequences,

$$x_i(n) \equiv x(2n+i),\ i = 0,\, 1; \quad X(z) = X_0(z^2) + z^{-1}X_1(z^2). \tag{4}$$

The *polyphase vector form* of a discrete-time signal is defined to be

$$\boldsymbol{x}(n) \equiv \begin{bmatrix} x_0(n) \\ x_1(n) \end{bmatrix}; \quad \boldsymbol{X}(z) \equiv \begin{bmatrix} X_0(z) \\ X_1(z) \end{bmatrix}. \tag{5}$$

The *analysis polyphase-with-advance representation* of a filter [4, equation (9)] is

$$f_j(n) \equiv f(2n-j),\; j = 0,\, 1; \quad F(z) = F_0(z^2) + zF_1(z^2).$$

Its *analysis polyphase vector representation* is

$$\boldsymbol{F}(z) \equiv \begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix} = \sum_{n=c}^{d} \boldsymbol{f}(n)\, z^{-n}, \tag{6}$$

$$\boldsymbol{f}(n) \equiv \begin{bmatrix} f_0(n) \\ f_1(n) \end{bmatrix} \quad \text{with } \boldsymbol{f}(c),\; \boldsymbol{f}(d) \neq \mathbf{0}. \tag{7}$$

Since we generally work with analysis filter bank representations, "polyphase" will mean "analysis polyphase-with-advance." The polyphase filter (6), (7) has the *polyphase support interval*

$$\text{supp\_int}(\boldsymbol{f}) \equiv [c, d], \tag{8}$$

which differs from the scalar support interval (2) for the same filter. The *polyphase order* of (6) is

$$\text{order}(\boldsymbol{F}) \equiv d - c\;. \tag{9}$$

These definitions generalize for FIR filter banks, $\{H_0(z),\, H_1(z)\}$. Decompose each filter $H_i(z)$ into its polyphase vector representation $\boldsymbol{H}_i(z)$ as in (6) and form the *polyphase matrix*

$$\mathbf{H}(z) \equiv \begin{bmatrix} \boldsymbol{H}_0^T(z) \\ \boldsymbol{H}_1^T(z) \end{bmatrix} = \sum_{n=c}^{d} \mathbf{h}(n)\, z^{-n}, \tag{10}$$

$$\mathbf{h}(n) \equiv \begin{bmatrix} \boldsymbol{h}_0^T(n) \\ \boldsymbol{h}_1^T(n) \end{bmatrix} \quad \text{with } \mathbf{h}(c),\; \mathbf{h}(d) \neq \mathbf{0}. \tag{11}$$

Bold italics denote column vectors and bold roman (upright) fonts denote matrices.

The polyphase support interval of the filter bank in (10), (11) is defined to be

$$\text{supp\_int}(\mathbf{h}) \equiv [c, d], \tag{12}$$

and the polyphase order is defined to be

$$\text{order}(\mathbf{H}) \equiv d - c. \tag{13}$$

With this notation, the output of the analysis bank in Figure 1 can be written

$$\boldsymbol{Y}(z) = \mathbf{H}(z)\boldsymbol{X}(z).$$

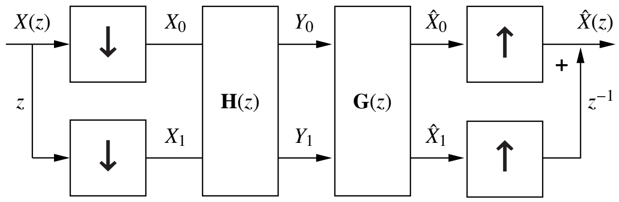

Fig. 2 The polyphase-with-advance representation of a two-channel multirate filter bank.

An analogous synthesis polyphase matrix representation, $\mathbf{G}(z)$, can be defined for the synthesis filter bank $\{G_0(z),\, G_1(z)\}$; see [4, Section II-A].

The block diagram for this matrix-vector filter bank representation, which we call the *polyphase-with-advance representation* [4], is shown in Figure 2. The polyphase representation transforms the non-translation-invariant analysis bank of Figure 1 into a demultiplex operation, $x(k) \mapsto \boldsymbol{x}(n)$, followed by a linear translation-invariant operator acting on vector-valued signals. The polyphase representation therefore reduces the study of multirate filter banks to the study of invertible transfer matrices over the Laurent polynomials.

Since Laurent *mono*mials are units, invertibility of $\mathbf{H}(z)$ over $\mathbb{C}[z,z^{-1}]$ is equivalent to

$$|\mathbf{H}(z)| \equiv \det \mathbf{H}(z) = \check{a}z^{-\check{d}}; \quad \check{a} \neq 0,\ \check{d} \in \mathbb{Z}. \tag{14}$$

$\check{d}$ is called the *determinantal delay* of $\mathbf{H}(z)$ and $\check{a}$ is called the *determinantal amplitude*. A filter bank satisfying (14) is called an *FIR perfect reconstruction (PR) filter bank* [24]. It was noted in [4, Theorem 1] that the family $\mathcal{F}$ of all FIR PR filter banks forms a nonabelian matrix group, called the *FIR filter bank group*. The *unimodular group*, $\mathcal{N}$, is the normal subgroup of $\mathcal{F}$ consisting of all matrices of determinant 1,

$$|\mathbf{H}(z)| = 1\,. \tag{15}$$

The unimodular group can also be regarded as $SL(2,\,\mathbb{C}\left[z,z^{-1}\right])$.

1.3 Linear Phase Filter Banks

It is easily shown [4, eqn. (20)] that a discrete-time signal is symmetric about one of its samples, $x(i_0)$, if and only if its polyphase vector representation (5) satisfies

$$\boldsymbol{X}(z^{-1}) = z^{i_0}\boldsymbol{\Lambda}(z)\boldsymbol{X}(z), \quad \text{where} \quad \boldsymbol{\Lambda}(z) \equiv \text{diag}(1,z^{-1}). \tag{16}$$

We say a signal satisfying (16) is *whole-sample symmetric* (WS) about $i_0 \in \mathbb{Z}$. Similarly, a discrete-time signal is *half-sample symmetric* (HS) about an odd multiple of 1/2 (indexed by $i_0 \in \mathbb{Z}/2$) if and only if

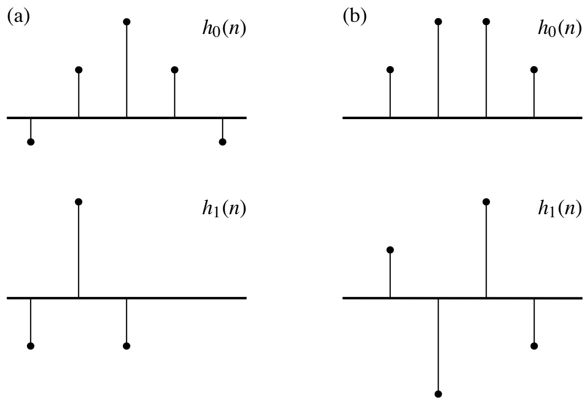

Fig. 3 (a) Whole-sample symmetric filter bank. (b) Half-sample symmetric filter bank.

$$\boldsymbol{X}(z^{-1}) = z^{(2i_0-1)/2}\mathbf{J}\boldsymbol{X}(z), \quad \text{where} \quad \mathbf{J} \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{17}$$

Analogous characterizations of whole- and half-sample *antisymmetry* (abbreviated WA and HA, respectively) are obtained by putting minus signs in (16) and (17). Real-valued discrete-time signals (or filters) possessing any of these symmetry properties are called *linear phase* signals (filters).

It was proven in [16] that the only *nontrivial* classes (classes with at least one nontrivial real degree of freedom) of two-channel FIR PR linear phase filter banks are the whole- and half-sample symmetric classes shown in Figure 3. Arbitrary combinations of symmetry are not necessarily compatible with invertibility; e.g., if both filters have odd lengths then both must be symmetric (WS). In an even-length filter bank, one filter must be symmetric (HS) while the other must be antisymmetric (HA). It was also proven in [16] that the sum of the impulse response lengths must be a multiple of 4, so it is possible for HS (but not WS) filter banks to have filters of *equal lengths*, as shown in Figure 3.

Linear phase properties of filter banks are also straightforward to characterize in the polyphase domain [4, Section III]. The *group delay* [17] of a linear phase FIR filter is equal to the midpoint (or axis of symmetry) of the filter's impulse response. Let d_i denote the group delay of h_i for $i = 0, 1$.

Lemma 1 ([4], Lemma 2). *A real-coefficient FIR transfer matrix* $\mathbf{H}(z)$ *is a WS analysis filter bank with group delays* d_0 *and* d_1 *if and only if*

$$\mathbf{H}(z^{-1}) = \text{diag}(z^{d_0}, z^{d_1})\mathbf{H}(z)\mathbf{\Lambda}(z^{-1}). \tag{18}$$

If $\mathbf{H}(z)$ satisfies (14) then the *delay-minimized WS filter bank* normalization

$$d_0 = 0,\ d_1 = -1 \tag{19}$$

ensures that the determinantal delay, $\check{d} = (d_0 + d_1 + 1)/2$, is zero and (18) becomes

$$\mathbf{H}(z^{-1}) = \mathbf{\Lambda}(z)\mathbf{H}(z)\mathbf{\Lambda}(z^{-1}). \tag{20}$$

The analogous delay-minimized HS filter bank normalization is

$$d_0 = -1/2 = d_1. \tag{21}$$

Both filters have the same axis of symmetry, as in Figure 3(b); we call such filter banks *concentric*. Delay-minimized HS filter banks are characterized by the relation

$$\mathbf{H}(z^{-1}) = \mathbf{L}\mathbf{H}(z)\mathbf{J} \quad \text{where} \quad \mathbf{L} \equiv \text{diag}(1,\, -1). \tag{22}$$

We now see a striking difference between the algebraic properties of WS and HS filter banks. Since $\mathbf{\Lambda}(z^{-1}) = \mathbf{\Lambda}^{-1}(z)$, (20) says that $\mathbf{\Lambda}(z)$ *intertwines* $\mathbf{H}(z)$ and $\mathbf{H}(z^{-1})$, so the set of all filter banks satisfying (20) (i.e., the set of all delay-minimized WS filter banks) forms a multiplicative group. In sharp contrast, filter banks satisfying (22) do *not* form a group.

Definition 1 ([1], Definition 8). The *unimodular WS group*, $\mathcal{W}$, is the group of all real FIR transfer matrices that satisfy both (15) and (20).

Definition 2 ([1], Definition 9). The *unimodular HS class*, $\mathfrak{H}$, is the set of all real FIR transfer matrices that satisfy both (15) and (22).

2 Lifting Factorization of Linear Phase Filter Banks

We now define lifting and apply it to linear phase filter banks, focusing on the problem of factoring linear phase filter banks into linear phase lifting steps.

2.1 Lifting Factorizations

Daubechies and Sweldens [9] used the Euclidean algorithm for $\mathbb{C}[z, z^{-1}]$ to prove that any unimodular FIR transfer matrix can be decomposed into a *lifting factorization* (or *lifting cascade*) of the form

$$\mathbf{H}(z) = \mathbf{D}_K\, \mathbf{S}_{N-1}(z) \cdots \mathbf{S}_1(z)\, \mathbf{S}_0(z)\ . \tag{23}$$

The diagonal matrix $\mathbf{D}_K \equiv \text{diag}(1/K,\, K)$ is a *unimodular gain-scaling matrix* with *scaling factor* $K \neq 0$. The lifting matrices $\mathbf{S}_i(z)$ are upper- or lower-triangular with ones on the diagonal and a lifting filter, $S_i(z)$, in the off-diagonal position.

In the factorization corresponding to Figure 4, the lifting matrix for the step $S_0(z)$ (which is a lowpass update) is upper-triangular, and the matrix for the second step (a highpass update) is lower-triangular. For example, the Haar filter bank

$$H_0(z) = (z+1)/2, \quad H_1(z) = z - 1, \tag{24}$$

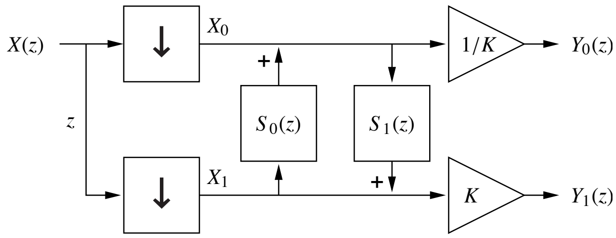

Fig. 4 Two-step lifting representation of a unimodular filter bank.

has a unimodular polyphase representation with two different lifting factorizations,

$$\mathbf{H}_{haar}(z) \equiv \begin{bmatrix} 1/2 & 1/2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{25}$$

$$= \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}. \tag{26}$$

Factorization (25) fits the ladder structure of Figure 4 with $S_0(z) = 1$, $S_1(z) = -1/2$, and $K = 2$. Factorization (26), on the other hand, begins with a *highpass* lifting update and does not require a gain-scaling operation.

Definition 3 ([13], Annex G). The *update characteristic* of a lifting step (or lifting matrix) is a binary flag, $m = 0$ or 1, indicating which polyphase channel is being updated by the lifting step.

For instance, the update characteristic, m_0, of the first lifting step in Figure 4 is "lowpass," coded with a zero ($m_0 = 0$), while the update characteristic of the second step is "highpass" ($m_1 = 1$). The update characteristic m_i is defined similarly for each matrix $\mathbf{S}_i(z)$ in a lifting cascade (23).

Next, we generalize (23) slightly to accommodate factorizations that lift one filter bank to another. A *partially factored lifting cascade*,

$$\mathbf{H}(z) = \mathbf{D}_K \, \mathbf{S}_{N-1}(z) \cdots \mathbf{S}_0(z) \, \mathbf{B}(z), \tag{27}$$

is an expansion relative to some *base* filter bank, $\mathbf{B}(z)$, with scalar filters $B_0(z)$ and $B_1(z)$. We sometimes write such factorizations in recursive form:

$$\begin{aligned} \mathbf{H}(z) &= \mathbf{D}_K \, \mathbf{E}^{(N-1)}(z), \\ \mathbf{E}^{(n)}(z) &= \mathbf{S}_n(z) \, \mathbf{E}^{(n-1)}(z), \quad 0 \le n < N, \\ \mathbf{E}^{(-1)}(z) &\equiv \mathbf{B}(z). \end{aligned} \tag{28}$$

2.2 Whole-Sample Symmetric Filter Banks

The fact that delay-minimized WS filter banks form a group makes it easy to characterize the lifting matrices that lift one delay-minimized WS filter bank to another,

$$\mathbf{F}(z) = \mathbf{S}(z)\,\mathbf{H}(z). \tag{29}$$

Lemma 2 ([4], Lemma 8). *A lifting matrix, $\mathbf{S}(z)$, lifts a filter bank satisfying (20) to another filter bank satisfying (20) if and only if $\mathbf{S}(z)$ also satisfies (20). An upper-triangular lifting matrix satisfies (20) if and only if its lifting filter is* half-*sample symmetric about $1/2$. A lower-triangular lifting matrix satisfies (20) if and only if its lifting filter is HS about $-1/2$.*

Note that HS lifting *filters* with appropriate group delays form lifting *matrices* that are WS filter banks. It is easy to show that the lifting filters symmetric about $1/2$ form an additive group, $\mathcal{P}_0$, of Laurent polynomials and that the upper-triangular lifting matrices with lifting filters in $\mathcal{P}_0$ form a multiplicative group, $\mathcal{U}$. Similarly, the lifting filters symmetric about $-1/2$ form an additive group, $\mathcal{P}_1$, and the lower-triangular lifting matrices with lifting filters in $\mathcal{P}_1$ form a multiplicative group, $\mathcal{L}$.

Given Lemma 2, it is natural to ask whether every filter bank in $\mathcal{W}$ has a lifting factorization of the form (23) in which every lifting matrix $\mathbf{S}_i(z)$ satisfies (20). The answer is yes, and the proof is a constructive, order-reducing recursion that does not rely on the Euclidean algorithm.

Theorem 1 ([4], Theorem 9). *A unimodular filter bank, $\mathbf{H}(z)$, satisfies the delay-minimized WS condition (20) if and only if it can be factored as*

$$\mathbf{H}(z) = \mathbf{D}_K\,\mathbf{S}_{N-1}(z)\cdots\mathbf{S}_1(z)\,\mathbf{S}_0(z), \tag{30}$$

where each lifting matrix, $\mathbf{S}_i(z)$, satisfies (20).

We refer to such decompositions as *WS group lifting factorizations.* This is the form of lifting factorizations specified in [13, Annex G] for user-defined WS filter banks.

Definition 1 of the unimodular WS group, $\mathcal{W}$, is independent of lifting, but we need lifting to define *reversible* WS filter banks. Let $\mathcal{U}_r$ and $\mathcal{L}_r$ be the subgroups of $\mathcal{U}$ and $\mathcal{L}$ with matrices whose lifting filters have *dyadic* coefficients of the form $k \cdot 2^n,\ k, n \in \mathbb{Z}$. Since gain-scaling operations are not generally invertible in fixed-precision arithmetic, gain scaling is not used in reversible implementations.

Definition 4 ([1], Example 3). The group $\mathcal{W}_r$ of reversible unimodular WS filter banks is defined to be the group of all transfer matrices $\mathbf{H}(z)$ generated by lifting factorizations (30) where $\mathbf{S}_i(z) \in \mathcal{U}_r \cup \mathcal{L}_r$ and $\mathbf{D}_K = \mathbf{I}$.

2.3 Half-Sample Symmetric Filter Banks

Lifting factorization of HS filter banks is harder (i.e., more interesting) than lifting factorization of WS filter banks, in part "because" HS filter banks do not form a group. For instance, the characterization in Lemma 2 of lifting matrices that lift one WS filter bank to another is equally valid for *left* lifts, as in (29), and *right* lifts in which $\mathbf{S}(z)$ acts on the right. This fails badly for HS filter banks.

Theorem 2 ([4], Theorem 12). *Suppose that $\mathbf{H}(z)$ is an HS filter bank satisfying the concentric delay-minimized condition (22). If $\mathbf{F}(z)$ is right-lifted from $\mathbf{H}(z)$,*

$$\mathbf{F}(z) = \mathbf{H}(z)\,\mathbf{S}(z),$$

then $\mathbf{F}(z)$ can only satisfy (22) if $\mathbf{S}(z) = \mathbf{I}$ and $\mathbf{F}(z) = \mathbf{H}(z)$.

Fortunately, half-sample symmetry can be preserved by left-lifting operations.

Lemma 3 ([4], Lemma 10). *If either $\mathbf{H}(z)$ or $\mathbf{F}(z)$ in (29) is an HS filter bank satisfying the concentric delay-minimized condition (22), then the other filter bank also satisfies (22) if and only if $\mathbf{S}(z)$ satisfies*

$$\mathbf{S}(z^{-1}) = \mathbf{L}\,\mathbf{S}(z)\,\mathbf{L} = \mathbf{S}^{-1}(z), \tag{31}$$

which says that the lifting filter is whole-sample antisymmetric (WA) about 0.

WA lifting filters form an additive group, $\mathcal{P}_a$, and the upper-triangular (resp., lower-triangular) lifting matrices with lifting filters in $\mathcal{P}_a$ form a group, $\mathcal{U}$ (resp., $\mathcal{L}$). In contrast to WS group lifting factorizations, concentric delay-minimized HS filter banks *never* factor completely into WA lifting steps [4, Theorem 13]. The obstruction, which does not exist for WS filter banks, is the possibility that a reduced-order intermediate HS filter bank in the factorization process will correspond to filters $H_0(z)$ and $H_1(z)$ of *equal lengths.* Given a concentric equal-length HS filter bank, it is *never* possible to reduce its order by factoring off a WA lifting step. This leaves us with an incomplete lifting theory for unimodular HS filter banks.

Theorem 3 ([4], Theorem 14). *A unimodular filter bank, $\mathbf{H}(z)$, satisfies the concentric delay-minimized HS convention (22) if and only if it can be decomposed into a partially factored lifting cascade of WA lifting steps satisfying (31) and a concentric equal-length HS base filter bank $\mathbf{B}(z)$ satisfying (22):*

$$\mathbf{H}(z) = \mathbf{S}_{N-1}(z)\cdots\mathbf{S}_0(z)\,\mathbf{B}(z). \tag{32}$$

There is no gain-scaling matrix, $\mathbf{D}_K$, in (32) since $\mathbf{B}(z)$ has been left unfactored.

One popular choice for the equal-length base filter bank in HS lifting constructions is the Haar filter bank, which has a particularly simple lifting factorization (26). The 2-tap/10-tap HS filter bank specified in JPEG 2000 Part 2 [13, Annex H.4.1.1.3] is lifted from the Haar via a lower-triangular 4th-order WA lifting step. Another important example is the 6-tap/10-tap HS filter bank in [13, Annex H.4.1.2.1]. This filter bank was originally constructed by spectral factorization and has a lifting factorization of the form $\mathbf{H}(z) = \mathbf{S}(z)\mathbf{B}(z)$, where $S(z)$ is a second-order WA filter and $\mathbf{B}(z)$ is an equal-length (6-tap/6-tap) HS filter bank.

Defining a class $\mathfrak{H}_r$ of *reversible* HS filter banks is awkward; see [1, Example 5].

3 Uniqueness of Linear Phase Lifting Factorizations

In the last section we saw that every filter bank in the unimodular WS and HS classes factors into linear phase lifting steps of an appropriate form. Lifting factorizations, like other elementary matrix decompositions, are highly nonunique, and although linear phase factorizations are more specialized than general lifting decompositions there seems little reason *a priori* to expect them to be unique. There are, however, a few trivial causes of nonuniqueness that we can exclude in an *ad hoc* fashion.

Definition 5 ([1], Definition 3). A lifting cascade (27) is *irreducible* if all lifting steps are nontrivial ($\mathbf{S}_i(z) \neq \mathbf{I}$) and there are no consecutive lifting matrices with the same update characteristic, i.e., the lifting matrices strictly alternate between lower- and upper-triangular.

Every lifting cascade can be simplified to an irreducible cascade using matrix multiplication. Merely restricting attention to irreducible lifting cascades is far from sufficient to ensure unique factorizations, as the two irreducible lifting factorizations of the Haar filter bank (25)–(26) show. To view nonuniqueness in a different light, move the lifting steps from (26) over to the right end of (25) and use [9, Section 7.3] to factor $\mathrm{diag}(1/2,\ 2)$ into lifting steps. This results in an irreducible lifting factorization of the identity,

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1/2 \\ 0 & 1 \end{bmatrix} . \tag{33}$$

In a similar manner, *any* transfer matrix with two distinct irreducible lifting factorizations gives rise to an irreducible factorization of the identity; cf. [1, Example 1], which presents an irreducible, reversible lifting factorization of the identity using linear phase (HS and HA) lifting filters. By constructing irreducible lifting factorizations of the identity, it is possible to sharpen the universal lifting factorization result of [9] into the following universal *nonunique* factorization result.

Proposition 1 ([1], Proposition 1). *If* $\mathbf{G}(z)$ *and* $\mathbf{H}(z)$ *are any FIR perfect reconstruction filter banks then* $\mathbf{G}(z)$ *can be irreducibly lifted from* $\mathbf{H}(z)$ *in infinitely many different ways.*

3.1 Group Lifting Structures

In light of the rich supply of elementary matrices, this plethora of irreducible lifting factorizations (almost all of which are useless for applications) results from our failure to specify precisely which liftings we regard as *useful*. The JPEG committee restricted the scope of [13, Annex G] to linear phase lifting factorizations of WS filter banks because these were considered to be the most useful liftings for conventional image coding, while [13, Annex H] was written to accommodate arbitrary lifted filter banks for niche applications. Taking a cue from the JPEG committee, we formalize a framework for specifying *restricted universes* of lifting factorizations. Group theory turns out to be a convenient tool for this task.

3.1.1 Lifting Matrix Groups

As mentioned above, upper-triangular (resp., lower-triangular) lifting matrices form multiplicative groups, $\mathcal{U}$ (resp., $\mathcal{L}$), as do lifting matrices whose lifting filters are restricted to additive groups of Laurent polynomials. This includes groups of filters whose symmetry and group delay are given, such as the groups $\mathcal{P}_0$ and $\mathcal{P}_1$ of HS lifting filters associated with Lemma 2. Define abelian group isomorphisms

$$\upsilon,\, \lambda : \mathbb{C}[z,z^{-1}] \to \mathcal{N}$$

that map a lifting filter $S(z) \in \mathbb{C}[z,z^{-1}]$ to lifting matrices,

$$\upsilon(S) \equiv \begin{bmatrix} 1 & S(z) \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \lambda(S) \equiv \begin{bmatrix} 1 & 0 \\ S(z) & 1 \end{bmatrix}. \tag{34}$$

Definition 6 ([1], Definition 4). Given two additive groups of Laurent polynomials, $\mathcal{P}_i < \mathbb{C}[z,z^{-1}]$, $i = 0,\, 1$, the groups $\mathcal{U} \equiv \upsilon(\mathcal{P}_0)$ and $\mathcal{L} \equiv \lambda(\mathcal{P}_1)$ are called the *lifting matrix groups* generated by $\mathcal{P}_0$ and $\mathcal{P}_1$.

3.1.2 Gain-Scaling Automorphisms

The unimodular gain-scaling matrices $\mathbf{D}_K \equiv \text{diag}(1/K, K)$ also form an abelian group with the product $\mathbf{D}_K\mathbf{D}_J = \mathbf{D}_{KJ}$, which says that we have an isomorphism

$$\mathbf{D} \colon \mathbb{R}^* \equiv \mathbb{R}\backslash\{0\} \xrightarrow{\cong} \mathcal{D} < \mathcal{N}. \tag{35}$$

$\mathcal{D}$ acts on $\mathcal{N}$ via inner automorphisms,

$$\gamma_K \mathbf{A}(z) \equiv \mathbf{D}_K\, \mathbf{A}(z)\, \mathbf{D}_K^{-1}\,, \qquad \gamma_K \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & K^{-2}b \\ K^2 c & d \end{bmatrix}. \tag{36}$$

This is equivalent to the intertwining relation

$$\mathbf{D}_K\,\mathbf{A}(z) = (\gamma_K \mathbf{A}(z))\,\mathbf{D}_K \tag{37}$$

and makes $\gamma\colon \mathbf{D}_K \mapsto \gamma_K$ a homomorphism of $\mathcal{D}$ onto a subgroup $\gamma(\mathcal{D}) < \mathrm{Aut}(\mathcal{N})$.

Definition 7 ([1], Definition 5). A group $\mathcal{G} < \mathcal{N}$ is $\mathcal{D}$-*invariant* if all of the inner automorphisms $\gamma_K \in \gamma(\mathcal{D})$ fix the group $\mathcal{G}$; i.e., $\gamma_K\mathcal{G} = \mathcal{G}$, so that $\gamma_K|_{\mathcal{G}} \in \mathrm{Aut}(\mathcal{G})$. This is equivalent to saying that $\mathcal{D}$ lies in the normalizer of $\mathcal{G}$ in $\mathcal{N}$:

$$\mathcal{D} \;<\; N_{\mathcal{N}}(\mathcal{G}) \;\equiv\; \left\{\mathbf{A} \in \mathcal{N} : \mathbf{A}\mathcal{G}\mathbf{A}^{-1} = \mathcal{G}\right\}\,.$$

For instance, when the lifting filter groups $\mathcal{P}_0$ and $\mathcal{P}_1$ are *vector spaces* it follows easily from (36) that $\mathcal{U} \equiv \upsilon(\mathcal{P}_0)$ and $\mathcal{L} \equiv \lambda(\mathcal{P}_1)$ are $\mathcal{D}$-invariant matrix groups.

3.1.3 Definition of Group Lifting Structures

We now have the machinery needed to define a "universe" of lifting factorizations. In the following, $\mathfrak{B}$ denotes a set (not necessarily a group) of base filter banks from which other filter banks are lifted in partially factored lifting cascades (27).

Definition 8 ([1], Definitions 6 and 7). A *group lifting structure* is an ordered four-tuple,

$$\mathfrak{S} \equiv (\mathcal{D},\,\mathcal{U},\,\mathcal{L},\,\mathfrak{B}),$$

where $\mathcal{D}$ is a gain-scaling group, $\mathcal{U}$ and $\mathcal{L}$ are upper- and lower-triangular lifting matrix groups, and $\mathfrak{B} \subset \mathcal{N}$. The *lifting cascade group*, $\mathcal{C}$, generated by $\mathfrak{S}$ is the subgroup of $\mathcal{N}$ generated by $\mathcal{U}$ and $\mathcal{L}$:

$$\mathcal{C} \equiv \langle \mathcal{U} \cup \mathcal{L} \rangle = \{\mathbf{S}_1 \cdots \mathbf{S}_k \colon k \geq 1,\ \mathbf{S}_i \in \mathcal{U} \cup \mathcal{L}\}\,. \tag{38}$$

The *scaled lifting group*, $\mathcal{S}$, generated by $\mathfrak{S}$ is the subgroup generated by $\mathcal{D}$ and $\mathcal{C}$:

$$\mathcal{S} \equiv \langle \mathcal{D} \cup \mathcal{C} \rangle = \{\mathbf{A}_1 \cdots \mathbf{A}_k \colon k \geq 1,\ \mathbf{A}_i \in \mathcal{D} \cup \mathcal{U} \cup \mathcal{L}\}\,. \tag{39}$$

We say $\mathfrak{S}$ is a $\mathcal{D}$-*invariant group lifting structure* if $\mathcal{U}$ and $\mathcal{L}$, and therefore $\mathcal{C}$, are $\mathcal{D}$-invariant groups.

Given a group lifting structure, the universe of all filter banks generated by $\mathfrak{S}$ is

$$\mathcal{D}\mathcal{C}\mathfrak{B} \equiv \{\mathbf{DCB} \colon \mathbf{D} \in \mathcal{D},\ \mathbf{C} \in \mathcal{C}\,,\ \mathbf{B} \in \mathfrak{B}\}\,.$$

The statement "$\mathbf{H}$ has a (group) lifting factorization in $\mathfrak{S}$" means $\mathbf{H} \in \mathcal{D}\mathcal{C}\mathfrak{B}$. $\mathbf{H}$ has a lifting factorization in $\mathfrak{S}$ if and only if it has an *irreducible* factorization in $\mathfrak{S}$.

The group lifting structure that characterizes the universe of WS group lifting factorizations is defined as follows. The lifting matrix groups $\mathcal{U} \equiv \upsilon(\mathcal{P}_0)$ and $\mathcal{L} \equiv \lambda(\mathcal{P}_1)$ are determined by the groups $\mathcal{P}_0$ and $\mathcal{P}_1$ of HS lifting filters defined in Section 2.2. By Theorem 1 unimodular WS filter banks factor completely over $\mathcal{U}$ and $\mathcal{L}$, so we set $\mathfrak{B} \equiv \{\mathbf{I}\}$. Since $\mathcal{P}_0$ and $\mathcal{P}_1$ are vector spaces, setting $\mathcal{D} \equiv \mathbf{D}(\mathbb{R}^*)$ results in a $\mathcal{D}$-invariant group lifting structure, $\mathfrak{S}_{\mathcal{W}} \equiv (\mathcal{D}, \mathcal{U}, \mathcal{L}, \mathfrak{B})$. The conclusion of Theorem 1 can be stated succinctly in terms of $\mathcal{C}_{\mathcal{W}} \equiv \langle \mathcal{U} \cup \mathcal{L} \rangle$ as

$$\mathcal{W} = \mathcal{D}\mathcal{C}_{\mathcal{W}}\mathfrak{B} = \mathcal{D}\mathcal{C}_{\mathcal{W}} . \tag{40}$$

The group lifting structure for delay-minimized HS lifting factorizations is more complicated. The lifting matrix groups $\mathcal{U} \equiv \upsilon(\mathcal{P}_a)$ and $\mathcal{L} \equiv \lambda(\mathcal{P}_a)$ are determined by the group $\mathcal{P}_a$ of WA lifting filters defined in Section 2.3. Per Theorem 3, we define $\mathfrak{B}_{\mathfrak{H}}$ to be the set of all concentric equal-length HS filter banks. Defining $\mathcal{D} \equiv \mathbf{D}(\mathbb{R}^*)$ results in a $\mathcal{D}$-invariant group lifting structure, $\mathfrak{S}_{\mathfrak{H}} \equiv (\mathcal{D}, \mathcal{U}, \mathcal{L}, \mathfrak{B}_{\mathfrak{H}})$. With $\mathcal{C}_{\mathfrak{H}} \equiv \langle U \cup L \rangle$ the conclusion of Theorem 3 can be stated as

$$\mathfrak{H} = \mathcal{D}\mathcal{C}_{\mathfrak{H}}\mathfrak{B}_{\mathfrak{H}} . \tag{41}$$

Group lifting structures $\mathfrak{S}_{\mathcal{W}_r}$ and $\mathfrak{S}_{\mathfrak{H}_r}$ for *reversible* WS and HS filter banks are defined in [1, Section IV].

3.2 Unique Irreducible Group Lifting Factorizations

We need one more hypothesis in addition to irreducibility to infer uniqueness of group lifting factorizations within a given group lifting structure. The key is found in the fact that nonunique lifting factorizations can be rewritten as irreducible lifting factorizations of the identity, such as (33). Given a (nonconstant) lifting of the identity like [1, equation (21)], if some partial product $\mathbf{E}^{(n)}(z)$ of lifting steps (28) has positive polyphase order then the order of subsequent partial products must eventually *decrease* because the final product, $\mathbf{I}$, has order zero. This suggests that lifting structures that only generate "order-increasing" cascades will generate *unique* factorizations, an idea that will be made rigorous in Theorem 4.

Definition 9 ([1], Definition 10). A lifting cascade (27) is *strictly polyphase order-increasing* (usually shortened to *order-increasing*) if the order (13) of each intermediate polyphase matrix (28) is strictly greater than that of its predecessor:

$$\text{order}\left(\mathbf{E}^{(n)}\right) > \text{order}\left(\mathbf{E}^{(n-1)}\right) \quad \text{for } 0 \le n < N.$$

A group lifting structure, $\mathfrak{S}$, is called order-increasing if every irreducible cascade in $\mathcal{C}\mathfrak{B}$ is order-increasing.

3.2.1 An Abstract Uniqueness Theorem

Theorem 4 ([1], Theorem 1). *Suppose that $\mathfrak{S}$ is a $\mathcal{D}$-invariant, order-increasing group lifting structure. Let $\mathbf{H}(z)$ be a transfer matrix generated by $\mathfrak{S}$, and suppose we are given two irreducible group lifting factorizations of $\mathbf{H}(z)$ in $\mathcal{D}\mathfrak{C}\mathfrak{B}$:*

$$\mathbf{H}(z) = \mathbf{D}_K\,\mathbf{S}_{N-1}(z)\cdots\mathbf{S}_0(z)\,\mathbf{B}(z) \tag{42}$$
$$= \mathbf{D}_{K'}\,\mathbf{S}'_{N'-1}(z)\cdots\mathbf{S}'_0(z)\,\mathbf{B}'(z)\ . \tag{43}$$

Then (42) and (43) satisfy the following three properties:

$$N' = N\,, \tag{44}$$
$$\mathbf{B}'(z) = \mathbf{D}_\alpha\,\mathbf{B}(z) \quad \textit{where } \alpha \equiv K/K', \tag{45}$$
$$\mathbf{S}'_i(z) = \gamma_\alpha\mathbf{S}_i(z) \quad \textit{for } i = 0,\ldots,N-1. \tag{46}$$

If, in addition, $\mathbf{B}(z)$ and $\mathbf{B}'(z)$ share a nonzero matrix entry at some point z_0 then the factorizations (42) and (43) are identical; *i.e., $K' = K$, $\mathbf{B}'(z) = \mathbf{B}(z)$, and*

$$\mathbf{S}'_i(z) = \mathbf{S}_i(z) \quad \textit{for } i = 0,\ldots,N-1. \tag{47}$$

It also follows that $K' = K$ if either of the scalar base filters, $B_0(z)$ or $B_1(z)$, shares a nonzero value with its primed counterpart; e.g., if the base filter banks have equal lowpass DC responses.

The relationship described by (44)–(46) leads to the following definition.

Definition 10 ([1], Definition 11). Two factorizations of $\mathbf{H}(z)$ that satisfy (44)–(46) are said to be *equivalent modulo rescaling.* If *all* irreducible group lifting factorizations of $\mathbf{H}(z)$ are equivalent modulo rescaling for *every* $\mathbf{H}(z)$ generated by $\mathfrak{S}$, we say that irreducible factorizations in $\mathfrak{S}$ are *unique modulo rescaling.*

3.2.2 Application to WS and HS Group Lifting Structures

Applying Theorem 4 is nontrivial, and verifying the order-increasing property is the hardest aspect of the whole theory. The key lemma for proving the order-increasing property for the WS and HS group lifting structures is the following result.

Lemma 4 ([2], Lemma 2). *Let $\mathfrak{S}$ be a group lifting structure satisfying the following two polyphase vector conditions.*

1. *For all $\mathbf{B}(z) \in \mathfrak{B}$, the polyphase support intervals (8) for the base polyphase filter vectors are equal:*

$$\text{supp\_int}(\boldsymbol{b}_0) = \text{supp\_int}(\boldsymbol{b}_1). \tag{48}$$

2. *For all irreducible lifting cascades in $\mathfrak{C}\mathfrak{B}$, the polyphase support intervals (8) for the intermediate polyphase filter vectors satisfy the proper inclusions*

$$\text{supp\_int}\left(\boldsymbol{e}^{(n)}_{1-m_n}\right) \subsetneq \text{supp\_int}\left(\boldsymbol{e}^{(n)}_{m_n}\right) \quad \textit{for } n \geq 0. \tag{49}$$

It then follows that $\mathfrak{S}$ is strictly polyphase order-increasing.

Hypothesis (48) is the correct answer to the ill-posed question, "What do all concentric equal-length HS base filter banks have in common with the lazy wavelet filter bank, **I**?" This was one of the last pieces of the uniqueness puzzle to be solved and unified the uniqueness proofs for the WS and HS cases.

Theorem 5 ([2], Theorem 1). *Let $\mathfrak{S}_{\mathcal{W}}$ and $\mathfrak{S}_{\mathcal{W}_r}$ be the group lifting structures defined in [1, Section IV-A]. Every filter bank in $\mathcal{W}$ has a unique irreducible lifting factorization in $\mathfrak{S}_{\mathcal{W}}$ and every filter bank in $\mathcal{W}_r$ has a unique irreducible lifting factorization in $\mathfrak{S}_{\mathcal{W}_r}$.*

Corollary 1 ([2], Corollary 1). *A delay-minimized unimodular WS filter bank can be specified in JPEG 2000 Part 2 Annex G syntax in one and only one way.*

The proof of Theorem 5 involves deriving the support-interval covering property (49) needed to invoke Lemma 4 and Theorem 4. The support-interval covering property results from the following tedious lemma based on the recursive formulation of lifting (28). The update characteristic of $\mathbf{S}_n(z)$ (Definition 3) is m_n and the *support radius* of a filter is the radius of its support interval,

$$\text{supp\_rad}(f) \equiv \left\lfloor \frac{b-a+1}{2} \right\rfloor, \quad \text{where} \quad [a,b] = \text{supp\_int}(f). \tag{50}$$

Lemma 5 ([2], Lemma 5). *Let $\mathbf{S}_{N-1}(z)\cdots\mathbf{S}_0(z) \in \mathcal{C}_{\mathcal{W}}$ be an irreducible cascade with intermediate scalar filters $E^{(n)}_i(z)$, $i = 0,\, 1$. Let $r^{(n)}_i$ be the support radius of $e^{(n)}_i$, and let $t^{(n)} \geq 1$ be the support radius of the HS lifting filter $S_n(z)$. Then* $\text{supp\_int}\left(e^{(n)}_i\right)$ *is centered at $-i$,*

$$\text{supp\_int}\left(e^{(n)}_i\right) = \left[-r^{(n)}_i - i,\, r^{(n)}_i - i\right], \quad i = 0,\, 1,$$

where

$$r^{(n)}_{m_n} = r^{(n)}_{1-m_n} + 2t^{(n)} - 1 \quad \textit{for } n \geq 0, \tag{51}$$

$$r^{(n)}_{1-m_n} = r^{(n-1)}_{m_n} + 2t^{(n-1)} - 1 \quad \textit{for } n \geq 1, \tag{52}$$

with $r^{(0)}_{1-m_0} = r^{(-1)}_{1-m_0} = 0.$

There is a similar unique factorization result for unimodular HS filter banks.

Theorem 6 ([2], Theorem 2). *Let $\mathfrak{S}_{\mathfrak{H}}$ and $\mathfrak{S}_{\mathfrak{H}_r}$ be the group lifting structures defined in [1, Section IV-B]. Every filter bank in $\mathfrak{H}$ has an irreducible group lifting factorization in $\mathfrak{S}_{\mathfrak{H}}$ that is unique modulo rescaling. Every filter bank in $\mathfrak{H}_r$ has a unique irreducible group lifting factorization in $\mathfrak{S}_{\mathfrak{H}_r}$.*

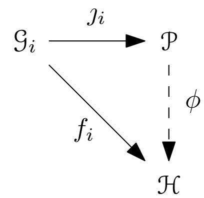

Fig. 5 Commutative diagram defining a free product of the groups $\mathcal{G}_i$.

4 Group-Theoretic Structure of Linear Phase Filter Banks

We can now characterize the group-theoretic structure of the groups generated by a $\mathcal{D}$-invariant, order-increasing group lifting structure. First we consider the lifting cascade group, $\mathcal{C}$, which only depends on $\mathcal{U}$ and $\mathcal{L}$, after which we consider the structure generated by scaling operations in the scaled lifting group, $\mathcal{S}$.

4.1 Free Product Structure of Lifting Cascade Groups

Recall the definition of free products in the category of groups.

Definition 11 ([11, 18]). Let $\{\mathcal{G}_i : i \in I\}$ be an indexed family of groups, and let $\mathcal{P}$ be a group with homomorphisms $\jmath_i : \mathcal{G}_i \to \mathcal{P}$. Then $\mathcal{P}$ is called a *free product of the groups* $\mathcal{G}_i$ if and only if, for every group $\mathcal{H}$ and family of homomorphisms $f_i : \mathcal{G}_i \to \mathcal{H}$, there exists a unique homomorphism $\phi : \mathcal{P} \to \mathcal{H}$ such that $\phi \circ \jmath_i = f_i$ for all $i \in I$. This is equivalent to saying that there exists a unique homomorphism ϕ such that the diagram in Figure 5 commutes for all $i \in I$.

Defining free products via the universal mapping property in Figure 5 means free products are *coproducts* in the category of groups and are therefore uniquely determined (up to isomorphism) by their generators $\mathcal{G}_i$ [11, Theorem I.7.5], [18, Theorem 11.50]. There is a constructive procedure (the "reduced word construction" [11, 18]) that generates a canonical realization of the free product of an arbitrary family of groups. Standard notation for free products is $\mathcal{P} = \mathcal{G}_1 * \mathcal{G}_2 * \cdots$.

The intuition behind Theorem 7 (below) is the identification of irreducible group lifting factorizations over $\mathcal{U}$ and $\mathcal{L}$ with the group of reduced words over the alphabet $\mathcal{U} \cup \mathcal{L}$, which is the canonical realization of $\mathcal{U} * \mathcal{L}$. The reduced word construction of $\mathcal{U} * \mathcal{L}$ is a somewhat technical chore when done rigorously, and it would be a messy affair at best to write down and verify an isomorphism between the group of reduced words over $\mathcal{U} \cup \mathcal{L}$ and a lifting cascade group in one-to-one correspondence with a collection of irreducible group lifting factorizations. For this reason the proof presented in [3] avoids the details of the reduced word construction and instead uses uniqueness of irreducible group lifting factorizations to show that $\mathcal{C}$ satisfies the categorical definition of a coproduct.

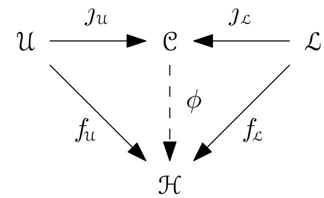

Fig. 6 Universal mapping property for the coproduct $\mathcal{C} \cong \mathcal{U} * \mathcal{L}$.

4.1.1 Lifting Cascade Groups are Free Products of $\mathcal{U}$ and $\mathcal{L}$

An easy lemma is needed to deal with group lifting structures whose irreducible group lifting factorizations are only unique modulo rescaling.

Lemma 6 ([3], Lemma 1). *If $(\mathcal{D}, \mathcal{U}, \mathcal{L}, \mathfrak{B})$ is a $\mathcal{D}$-invariant, order-increasing group lifting structure with lifting cascade group $\mathcal{C} \equiv \langle \mathcal{U} \cup \mathcal{L} \rangle$ then irreducible group lifting factorizations in $\mathcal{C}$ are unique, even if irreducible group lifting factorizations of filter banks in $\mathcal{D}\mathcal{C}\mathfrak{B}$ are only unique modulo rescaling.*

Lemma 6 ensures that all $\mathcal{D}$-invariant, order-increasing group lifting structures satisfy the hypotheses of the following theorem, whose proof consists of showing that $\mathcal{C}$ satisfies the universal mapping property in Figure 6.

Theorem 7 ([3], Theorem 1). *Let $\mathcal{U}$ and $\mathcal{L}$ be upper- and lower-triangular lifting matrix groups with lifting cascade group $\mathcal{C} \equiv \langle \mathcal{U} \cup \mathcal{L} \rangle$. If every element of $\mathcal{C}$ has a unique irreducible group lifting factorization over $\mathcal{U} \cup \mathcal{L}$ then $\mathcal{C}$ is isomorphic to the free product of $\mathcal{U}$ and $\mathcal{L}$:*

$$\mathcal{C} \cong \mathcal{U} * \mathcal{L} .$$

This free product structure, $\mathcal{C} \cong \mathcal{U} * \mathcal{L}$, is one of the conditions that are required for $\mathcal{C}$ to be a free group.

Theorem 8 ([3], Theorem 2). *Let $\mathcal{C} \equiv \langle \mathcal{U} \cup \mathcal{L} \rangle$ be a lifting cascade group over nontrivial lifting matrix groups $\mathcal{U}$ and $\mathcal{L}$. $\mathcal{C}$ is a free group (necessarily on two generators) if and only if $\mathcal{U}$ and $\mathcal{L}$ are infinite cyclic groups and $\mathcal{C} \cong \mathcal{U} * \mathcal{L}$.*

4.2 Semidirect Product Structure of Scaled Lifting Groups

Consider the interaction between the gain-scaling group $\mathcal{D}$ and the lifting cascade group $\mathcal{C}$ in a scaled lifting group, $\mathcal{S} \equiv \langle \mathcal{D} \cup \mathcal{C} \rangle$. As we have seen, $\mathcal{D}$ acts on $\mathcal{C}$ via inner automorphisms so it is not surprising that, under suitable hypotheses, $\mathcal{S}$ has the structure of a semidirect product, whose definition we now review.

Definition 12 ([14, 11, 18]). Let $\mathcal{G}$ be a (multiplicative) group with identity element $1_{\mathcal{G}}$ and subgroups $\mathcal{K}$ and $\mathcal{Q}$. $\mathcal{G}$ is an *(internal) semidirect product of $\mathcal{K}$ by $\mathcal{Q}$*, denoted $\mathcal{G} = \mathcal{Q} \ltimes \mathcal{K}$, if the following three axioms are satisfied.

$$\mathcal{G} = \langle \mathcal{K} \cup \mathcal{Q} \rangle \quad (\mathcal{K} \text{ and } \mathcal{Q} \text{ generate } \mathcal{G}) \tag{53}$$

$$\mathcal{K} \lhd \mathcal{G} \quad (\mathcal{K} \text{ is a normal subgroup of } \mathcal{G}) \tag{54}$$

$$\mathcal{K} \cap \mathcal{Q} = 1_{\mathcal{G}} \quad (\text{the trivial group}) \tag{55}$$

If $\mathcal{G} = \mathcal{Q} \ltimes \mathcal{K}$ then $\langle \mathcal{K} \cup \mathcal{Q} \rangle = \mathcal{Q}\mathcal{K}$ and such product representations, $g = qk$ for $g \in \mathcal{G} = \mathcal{Q}\mathcal{K}$, are unique.

For groups $\mathcal{K}$ and $\mathcal{Q}$ that are not subgroups of a common parent, a similar construction called an *external semidirect product*, denoted $\mathcal{G} = \mathcal{Q} \ltimes_\theta \mathcal{K}$, can be performed whenever we have an automorphic group action $\theta\colon \mathcal{Q} \to \mathrm{Aut}(\mathcal{K})$.

4.2.1 Scaled Lifting Groups are Semidirect Products of $\mathcal{C}$ by $\mathcal{D}$

Let $\mathfrak{S} = (\mathcal{D}, \mathcal{U}, \mathcal{L}, \mathfrak{B})$ be a group lifting structure with lifting cascade group $\mathcal{C}$ and scaled lifting group $\mathcal{S}$. The following theorem has the same hypotheses as those of Theorem 4, but rather than invoking the unique factorization theorem the argument in [3] proves Theorem 9 directly from the hypotheses.

Theorem 9 ([3], Theorem 3). *If $\mathfrak{S}$ is a $\mathcal{D}$-invariant, order-increasing group lifting structure then $\mathcal{S}$ is the internal semidirect product of $\mathcal{C}$ by $\mathcal{D}$:*

$$\mathcal{S} = \mathcal{D} \ltimes \mathcal{C}.$$

This result can be combined with Theorem 7 to yield a complete group-theoretic description of the group of unimodular WS filter banks,

$$\mathcal{W} = \mathcal{S}_{\mathcal{W}} = \mathcal{D}\mathcal{C}_{\mathcal{W}}.$$

Corollary 2 ([3], Corollary 2). *Let $\mathfrak{S}_{\mathcal{W}} \equiv (\mathcal{D}, \mathcal{U}, \mathcal{L}, \mathbf{I})$ be the group lifting structure for the unimodular WS group, $\mathcal{W}$, defined in [1, Section IV]. The group-theoretic structure of $\mathcal{W}$ is*

$$\mathcal{W} \cong \mathcal{D} \ltimes_\theta (\mathcal{U} * \mathcal{L}).$$

A similar characterization is possible for HS filter banks. While $\mathfrak{H}$ is not a group, the product representation

$$\mathfrak{H} = \mathcal{D}\mathcal{C}_{\mathfrak{H}}\mathfrak{B}_{\mathfrak{H}} = \mathcal{S}_{\mathfrak{H}}\mathfrak{B}_{\mathfrak{H}}, \tag{56}$$

$$\mathfrak{B}_{\mathfrak{H}} \equiv \{\mathbf{B} \in \mathfrak{H} \colon \text{order}(B_0) = \text{order}(B_1)\}, \tag{57}$$

exhibits $\mathfrak{H}$ as a collection of *right cosets*, $\mathcal{S}_{\mathfrak{H}}\mathbf{B}$, of $\mathcal{S}_{\mathfrak{H}}$ by elements of $\mathfrak{B}_{\mathfrak{H}}$. These cosets do not *partition* $\mathfrak{H}$, however, since they are not disjoint: $\mathbf{B}' \equiv \mathbf{D}_\alpha \mathbf{B} \in \mathfrak{B}_{\mathfrak{H}}$ implies $\mathcal{S}_{\mathfrak{H}}\mathbf{B} = \mathcal{S}_{\mathfrak{H}}\mathbf{B}'$. To obtain a nonredundant partition of $\mathfrak{H}$ into cosets, we can either eliminate scaling matrices (i.e., form cosets of $\mathcal{C}_{\mathfrak{H}}$ rather than of $\mathcal{S}_{\mathfrak{H}}$) or else normalize the elements of $\mathfrak{B}_{\mathfrak{H}}$.

Corollary 3 ([3], Corollary 3). *Let $\mathfrak{S}_{\mathfrak{H}} \equiv (\mathcal{D}, \mathcal{U}, \mathcal{L}, \mathfrak{B}_{\mathfrak{H}})$ be the group lifting structure for the unimodular HS class, $\mathfrak{H}$, defined in [1, Section IV]. The group-theoretic structure of $\mathcal{S}_{\mathfrak{H}}$ is*

$$\mathcal{S}_{\mathfrak{H}} \cong \mathcal{D} \ltimes_{\theta} (\mathcal{U} * \mathcal{L}),$$

and $\mathfrak{H}$ can be partitioned into disjoint right cosets (but not left cosets) of either $\mathcal{C}_{\mathfrak{H}}$ or $\mathcal{S}_{\mathfrak{H}}$:

$$\mathfrak{H} = \bigcup \{\mathcal{C}_{\mathfrak{H}} \mathbf{B} \colon \mathbf{B} \in \mathfrak{B}_{\mathfrak{H}}\} \tag{58}$$

$$= \bigcup \{\mathcal{S}_{\mathfrak{H}} \mathbf{B} \colon \mathbf{B} \in \mathfrak{B}'_{\mathfrak{H}}\}, \tag{59}$$

where $\mathfrak{B}'_{\mathfrak{H}}$ is given by, e.g.,

$$\mathfrak{B}'_{\mathfrak{H}} \equiv \{\mathbf{B} \in \mathfrak{B}_{\mathfrak{H}} \colon B_0(1) = 1\}. \tag{60}$$

Scaled lifting groups with the structure $\mathcal{S} \cong \mathcal{D} \ltimes_{\theta} (\mathcal{U} * \mathcal{L})$ have formal similarities [3, Section IV] to other examples in the mathematical literature of continuous groups with dilations, such as *homogeneous groups* [10, 20]. Unlike homogeneous groups, however, scaled lifting groups are neither nilpotent nor finite-dimensional, so scaled lifting groups at present appear to be a new addition to the realm of continuous groups with scaling automorphisms.

5 Conclusions

We have surveyed recent results characterizing the group-theoretic structure of the two principal classes of two-channel linear phase perfect reconstruction unimodular filter banks, the whole-sample symmetric and the half-sample symmetric classes. WS filter banks presented in the polyphase-with-advance representation naturally form a multiplicative subgroup, $\mathcal{W}$, of the group of all unimodular matrix Laurent polynomials. Although the class $\mathfrak{H}$ of unimodular HS filter banks does not form a group, lifting factorization theory shows that HS filter banks form cosets of a particular group generated by unimodular diagonal gain-scaling matrices and lifting matrices with whole-sample antisymmetric lifting filters. An algebraic framework known as a group lifting structure has been introduced for formalizing the group-theoretic structure of lifting factorizations, and it has been shown that the group lifting structures for WS (respectively, HS) filter banks satisfy a nontrivial polyphase order-increasing property that implies uniqueness of irreducible group lifting factorizations.

These unique factorization results have in turn been used to characterize the structure (up to isomorphism) of the lifting cascade group and the scaled lifting group associated with each of these classes of linear phase filter banks. Specifically, in both cases the lifting cascade group generated by the linear phase lifting matrices is the free product of the upper- and lower-triangular lifting matrix groups, $\mathcal{C} \cong \mathcal{U} * \mathcal{L}$. Also in both cases, the scaled lifting group generated by the lifting cascade group and the diagonal gain-scaling matrix group has the structure of a semidirect product, $\mathcal{S} = \mathcal{D}\mathcal{C} \cong \mathcal{D} \ltimes_{\theta} (\mathcal{U}*\mathcal{L})$. In the case of WS filter banks this directly furnishes the structure of the unimodular WS group, $\mathcal{W}$, since $\mathcal{W} = \mathcal{S}_{\mathcal{W}}$. In the case of HS filter banks, $\mathfrak{H}$ is partitioned by the family of all right cosets of $\mathcal{C}_{\mathfrak{H}}$ by concentric equal-length base HS filter banks. Alternatively, $\mathfrak{H}$ is also partitioned by the family of all right cosets of $\mathcal{S}_{\mathfrak{H}}$ by concentric equal-length base HS filter banks with unit lowpass DC response.

Acknowledgements The original research papers [4, 1, 2, 3] described in this article were supported by the Los Alamos Laboratory-Directed Research & Development Program. Preparation of this article was supported by the DOE Office of Science and Kristi D. and Reilly R. Brislawn. The author also thanks the producers of the Ipe drawing editor (`http://ipe7.sourceforge.net`) and the TEXLive/MacTEX distribution (`http://www.tug.org/mactex`).

References

1. Brislawn, C.M.: Group lifting structures for multirate filter banks I: Uniqueness of lifting factorizations. IEEE Trans. Signal Process. **58**(4), 2068–2077 (2010). DOI 10.1109/TSP.2009.2039816. URL `http://dx.doi.org/10.1109/TSP.2009.2039816`
2. Brislawn, C.M.: Group lifting structures for multirate filter banks II: Linear phase filter banks. IEEE Trans. Signal Process. **58**(4), 2078–2087 (2010). DOI 10.1109/TSP.2009.2039818. URL `http://dx.doi.org/10.1109/TSP.2009.2039818`
3. Brislawn, C.M.: Group-theoretic structure of linear phase multirate filter banks. Tech. Rep. LA-UR-12-20858, Los Alamos National Lab (2012). URL `http://viz.lanl.gov/paper.html`. Submitted for publication
4. Brislawn, C.M., Wohlberg, B.: The polyphase-with-advance representation and linear phase lifting factorizations. IEEE Trans. Signal Process. **54**(6), 2022–2034 (2006). DOI 10.1109/TSP.2006.872582. URL `http://dx.doi.org/10.1109/TSP.2006.872582`
5. Bruekers, F.A.M.L., van den Enden, A.W.M.: New networks for perfect inversion and perfect reconstruction. IEEE J. Selected Areas Commun. **10**(1), 129–137 (1992)
6. Calderbank, A.R., Daubechies, I., Sweldens, W., Yeo, B.L.: Wavelet transforms that map integers to integers. Applied and Computational Harmonic Analysis **5**(3), 332–369 (1998). DOI 10.1006/acha.1997.0238. URL `http://www.sciencedirect.com/science/article/B6WB3-45KKTPF-H/2/489d06bc0099f6a398eb534b2a8a8481`
7. Crochiere, R.E., Rabiner, L.R.: Multirate Digital Signal Processing. Prentice Hall, Englewood Cliffs, NJ (1983)
8. Daubechies, I.C.: Ten Lectures on Wavelets. No. 61 in CBMS-NSF Regional Conf. Series in Appl. Math., (Univ. Mass.—Lowell, June 1990). Soc. Indust. Appl. Math., Philadelphia (1992)
9. Daubechies, I.C., Sweldens, W.: Factoring wavelet transforms into lifting steps. J. Fourier Anal. Appl. **4**(3), 245–267 (1998)
10. Folland, G.B., Stein, E.M.: Hardy Spaces on Homogeneous Groups. No. 28 in Princeton Mathematical Notes. Princeton Univ. Press, Princeton, NJ (1982)
11. Hungerford, T.W.: Algebra. Springer-Verlag, New York, NY (1974)
12. Information technology—JPEG 2000 image coding system, Part 1, ISO/IEC Int'l. Standard 15444-1, ITU-T Rec. T.800. Int'l. Org. Standardization (2000)
13. Information technology—JPEG 2000 image coding system, Part 2: Extensions, ISO/IEC Int'l. Standard 15444-2, ITU-T Rec. T.801. Int'l. Org. Standardization (2004)
14. MacLane, S., Birkhoff, G.: Algebra. Macmillan, New York, NY (1967)

15. Mallat, S.: A Wavelet Tour of Signal Processing, 2nd edn. Academic Press, San Diego, CA (1999)
16. Nguyen, T.Q., Vaidyanathan, P.P.: Two-channel perfect-reconstruction FIR QMF structures which yield linear-phase analysis and synthesis filters. Acoustics, Speech and Signal Processing, IEEE Transactions on **37**(5), 676–690 (1989)
17. Oppenheim, A.V., Schafer, R.W., Buck, J.R.: Discrete-Time Signal Processing, 2 edn. Prentice Hall, Upper Saddle River, NJ (1998)
18. Rotman, J.J.: An Introduction to the Theory of Groups, 4 edn. Springer-Verlag, New York, NY (1995)
19. Said, A., Pearlman, W.A.: Reversible image compression via multiresolution representation and predictive coding. In: Visual Commun. & Image Proc., *Proc. SPIE*, vol. 2094, pp. 664–674. SPIE, Cambridge, MA (1993)
20. Stein, E.M.: Harmonic Analysis. Princeton Univ. Press, Princeton, NJ (1993)
21. Strang, G., Nguyen, T.: Wavelets and Filter Banks. Wellesley-Cambridge, Wellesley, MA (1996)
22. Sweldens, W.: The lifting scheme: a custom-design construction of biorthogonal wavelets. Appl. Comput. Harmonic Anal. **3**(2), 186–200 (1996)
23. Sweldens, W.: The lifting scheme: a construction of second generation wavelets. SIAM J. Math. Anal. **29**(2), 511–546 (1998)
24. Vaidyanathan, P.P.: Multirate Systems and Filter Banks. Prentice Hall, Englewood Cliffs, NJ (1993)
25. Vetterli, M., Kovačević, J.: Wavelets and Subband Coding. Prentice Hall, Englewood Cliffs, NJ (1995)
26. Zandi, A., Allen, J.D., Schwartz, E.L., Boliek, M.: Compression with reversible embedded wavelets. In: Proc. Data Compress. Conf., pp. 212–221. IEEE Computer Soc., Snowbird, UT (1995)