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Demystification of the Geometric Fourier Transforms and resulting Convolution Theorems

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As it will turn out in this paper, the recent hype about most of the Clifford Fourier transforms is not thoroughly worth the pain. Almost every one that has a real application is separable and these transforms can be decomposed into a sum of real valued transforms with constant multivector factors. This fact makes their interpretation, their analysis, and their implementation almost trivial.

Keywords: geometric algebra, Clifford algebra, Fourier transform, trigonometric transform, convolution theorem

1. Introduction

For a general introduction to noncommutative harmonic analysis, see [1]. In recent literature, three different approaches to *hypercomplex* Fourier transforms have been considered. As in [2], we can identify them as follows:

- A: Eigenfunction approach
- B: Generalized roots of -1 approach
- C: Characters of spin group approach

Approach A is studied in many papers like [3, 4, 5], approach B comprehensively in [6]. The third approach is followed in [7, 8]. In this paper, we analyze the generalized square roots of -1 approach, because even though the concept of approach C differs very much from approach B, the resulting transforms can be expressed as special cases of approach B. This paper is self-consistent, but we highly recommend to take a look at [6] for a deeper understanding of the origin of the analyzed issues. This is where the following definition was introduced.

Definition 1 (Geometric Fourier transform) *The general **geometric Fourier transform** (GFT) $\mathscr{F}_{F_1,F_2}(\mathbf{A})$ of a multivector field $\mathbf{A}:\mathbb{R}^m=\mathbb{R}^{p',q'}\to C\ell_{p,q}, p'+q'=m\in\mathbb{N}, p+q=n\in\mathbb{N}$ is defined by the calculation rule*

$$\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) := \int_{\mathbb{R}^m} \prod_{f\in F_1} e^{-f(\mathbf{x},\mathbf{u})}\mathbf{A}(\mathbf{x}) \prod_{f\in F_2} e^{-f(\mathbf{x},\mathbf{u})}\,\mathrm{d}^m\mathbf{x}, \tag{1}$$

with two ordered finite sets $F_1=\{f_1(\mathbf{x},\mathbf{u}),...,f_\mu(\mathbf{x},\mathbf{u})\}$, $F_2=\{f_{\mu+1}(\mathbf{x},\mathbf{u}),...,f_\nu(\mathbf{x},\mathbf{u})\}$ of mappings $f_l(\mathbf{x},\mathbf{u}):\mathbb{R}^m\times\mathbb{R}^m\to\mathscr{I}^{p,q},\forall l=1,...,\nu$ and $\mathbf{x},\mathbf{u}\in\mathbb{R}^m$.

We will only work with transforms from Definition 1 throughout this paper. That covers all known Fourier transforms over Clifford algebras that belong to approach B and C.

Example 2 *Depending on the choice of F_1 and F_2 in the definition of the GFT, we get already developed transforms. The following six examples have already been chosen as standard examples in [6]. We will use them again to visualize how the general propositions reduce to special applications.*

1. *For $A:\mathbb{R}^n\to C\ell_{n,0}, n=2 \pmod 4$ or $n=3 \pmod 4$, we can reproduce the Clifford Fourier transform introduced by Jancewicz [9] for $n=3$ and expanded by Ebling and Scheuermann [10] for $n=2$ and Hitzer and Mawardi [11] for $n=2 \pmod 4$ or $n=3 \pmod 4$*

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using the configuration

$$\begin{aligned} F_1 =&\emptyset, \\ F_2 =&\{f_1\}, \\ f_1(\mathbf{x},\mathbf{u}) =&2\pi i_n \mathbf{x}\cdot\mathbf{u}, \end{aligned} \tag{2}$$

with i_n being the pseudoscalar of $C\ell_{n,0}$.

2. *Choosing multivector fields $\mathbb{R}^n \to C\ell_{0,n}$,*

$$\begin{aligned} F_1 =&\emptyset, \\ F_2 =&\{f_1,\dots,f_n\}, \\ f_k(\mathbf{x},\mathbf{u}) =&2\pi \mathbf{e}_k x_k u_k, \forall k=1,\dots,n, \end{aligned} \tag{3}$$

we have the Clifford Fourier transform introduced by Sommen [12] and reestablished by Bülow [13].

3. *For $A:\mathbb{R}^2 \to C\ell_{0,2} \approx \mathbb{H}$, and pure unit quaternions $\boldsymbol{i},\boldsymbol{j} \in \mathbb{H}$, $\boldsymbol{i}^2=\boldsymbol{j}^2=-1$, the quaternionic Fourier transform established by Ell [14], reestablished by Bülow [13] and later extended by Hitzer and Sangwine [15] to an arbitrary steerable (even non-orthonormal) pair of pure quaternion square roots of -1 is generated by*

$$\begin{aligned} F_1 =&\{f_1\}, \\ F_2 =&\{f_2\}, \\ f_1(\mathbf{x},\mathbf{u}) =&2\pi \boldsymbol{i} x_1 u_1, \\ f_2(\mathbf{x},\mathbf{u}) =&2\pi \boldsymbol{j} x_2 u_2. \end{aligned} \tag{4}$$

4. *Using $C\ell_{3,1}$, we can build the spacetime Fourier transform by Hitzer [16] with the $C\ell_{3,1}$-pseudoscalar i_4 and $\varepsilon_4=\pm 1$ as follows*

$$\begin{aligned} F_1 =&\{f_1\}, \\ F_2 =&\{f_2\}, \\ f_1(\mathbf{x},\mathbf{u}) =&\mathbf{e}_4 x_4 u_4, \\ f_2(\mathbf{x},\mathbf{u}) =&\varepsilon_4 \mathbf{e}_4 i_4 (x_1u_1+x_2u_2+x_3u_3). \end{aligned} \tag{5}$$

5. *The Clifford Fourier transform for color images by Batard, Berthier and Saint-Jean [7] for $m=2, n=4, A:\mathbb{R}^2 \to C\ell_{4,0}$, a fixed bivector B, and the pseudoscalar i can be written as*

$$\begin{aligned} F_1 =&\{f_1,f_2\}, \\ F_2 =&\{f_3,f_4\}, \\ f_1(\mathbf{x},\mathbf{u}) =&\frac{1}{2}(x_1u_1+x_2u_2)B, \\ f_2(\mathbf{x},\mathbf{u}) =&\frac{1}{2}(x_1u_1+x_2u_2)iB, \\ f_3(\mathbf{x},\mathbf{u}) =&-\frac{1}{2}(x_1u_1+x_2u_2)B, \\ f_4(\mathbf{x},\mathbf{u}) =&-\frac{1}{2}(x_1u_1+x_2u_2)iB. \end{aligned} \tag{6}$$

6. *For $C\ell_{0,n}$, the definition produces the cylindrical Fourier transform as introduced by Brackx, de Schepper and Sommen in [17] by*

$$\begin{aligned} F_1 =&\{f_1\}, \\ F_2 =&\emptyset, \\ f_1(\mathbf{x},\mathbf{u}) =&-\mathbf{x}\wedge\mathbf{u}. \end{aligned} \tag{7}$$

In the following section, we will introduce a powerful tool for the analysis of the geometric Fourier transforms: the trigonometric transform. Then, we will first use it to reveal the true nature of the subclasses of the separable and the explicitly invertible GFTs, which cover almost every applied Clifford Fourier transform. In the last two sections of the paper, we will enjoy the advantages of this insight and show how it leads to a convolution theorem that is superior to the one in [18] because it requires less restrictions.

2. The Trigonometric Transform

For the GFT convolution theorem in [18], we made use of geometric trigonometric transforms. These are generalized GFTs that may also contain the cosine of the functions F_1, F_2 or their sine paired with the square root of minus one instead of only their exponentials. In contrast to the hypercomplex definition in [18], we want to define real valued general trigonometric transforms, that only consist of scalar appearances of sines and cosines. Therefore we use the following notation.

Notation 3 *Let $C\ell_{p,q}$ denote the real Clifford algebra of the non-Euclidean vector space $\mathbb{R}^{p,q}$ [19]. A multivector valued function $f(\mathbf{x},\mathbf{u}):\mathbb{R}^2\to\mathscr{I}_{p,q}=\{B\in C\ell_{p,q},B^2\in\mathbb{R}^-\}\subset C\ell_{p,q}$ that squares to a negative real number, can be separated into a unit part and a real part. This separation is not unique. There are two functions*[†] $\pm i(\mathbf{x},\mathbf{u}):\mathbb{R}^2\to\{B\in C\ell_{p,q},B^2=-1\}\subset I_{p,q}\subset C\ell_{p,q}$ *that square to minus one*

$$\pm i(\mathbf{x},\mathbf{u})=\pm\frac{f(\mathbf{x},\mathbf{u})}{||f(\mathbf{x},\mathbf{u})||}, \tag{8}$$

and leave the positive real valued function $||f(\mathbf{x},\mathbf{u})||=\sqrt{-f(\mathbf{x},\mathbf{u})^2}:\mathbb{R}^2\to\mathbb{R}^+$. We can choose one of them and define the function $|f(\mathbf{x},\mathbf{u})|:\mathbb{R}^2\to\mathbb{R}$, which can also take negative values, by

$$|f(\mathbf{x},\mathbf{u})|=\frac{f(\mathbf{x},\mathbf{u})}{i(\mathbf{x},\mathbf{u})}. \tag{9}$$

Now, with $j\in\{0,1\}$, we define the decomposition

$$e_j^{f(\mathbf{x},\mathbf{u})}:=\begin{cases}\cos(|f(\mathbf{x},\mathbf{u})|), & \text{if } j=0,\\ \sin(|f(\mathbf{x},\mathbf{u})|), & \text{if } j=1.\end{cases} \tag{10}$$

Lemma 4 *The exponential of a multivector valued function $f(\mathbf{x},\mathbf{u}):\mathbb{R}^2\to\mathscr{I}_{p,q}=\{B\in C\ell_{p,q},B^2\in\mathbb{R}^-\}\subset C\ell_{p,q}$ that squares to a negative real number satisfies*

$$e^{f(\mathbf{x},\mathbf{u})}=\sum_{j\in\{0,1\}}\Big(\frac{f(\mathbf{x},\mathbf{u})}{|f(\mathbf{x},\mathbf{u})|}\Big)^j e_j^{f(\mathbf{x},\mathbf{u})}. \tag{11}$$

Proof: Because of $i(\mathbf{x},\mathbf{u})=\frac{f(\mathbf{x},\mathbf{u})}{|f(\mathbf{x},\mathbf{u})|}$ squares to -1 and $|f(\mathbf{x},\mathbf{u})\in\mathbb{R}$, the hypercomplex equivalent to the Euler equation as in [20] leads to

$$\begin{aligned}e^{f(\mathbf{x},\mathbf{u})} &= e^{i(\mathbf{x},\mathbf{u})|f(\mathbf{x},\mathbf{u})|}\\ &= \cos(|f(\mathbf{x},\mathbf{u})|)+i(\mathbf{x},\mathbf{u})\sin(|f(\mathbf{x},\mathbf{u})|)\\ &\overset{\text{Not. 3}}{=} i(\mathbf{x},\mathbf{u})^0 e_0^{f(\mathbf{x},\mathbf{u})}+i(\mathbf{x},\mathbf{u})^1 e_1^{f(\mathbf{x},\mathbf{u})}\\ &= \sum_{j\in\{0,1\}} i(\mathbf{x},\mathbf{u})^j e_j^{|f(\mathbf{x},\mathbf{u})|}\\ &= \sum_{j\in\{0,1\}}\Big(\frac{f(\mathbf{x},\mathbf{u})}{|f(\mathbf{x},\mathbf{u})|}\Big)^j e_j^{f(\mathbf{x},\mathbf{u})},\end{aligned} \tag{12}$$

which proves the assertion. □

Remark 5 *The minus sign that appears in the Fourier transform can be left with the real valued function*

$$\begin{aligned}e^{-f(\mathbf{x},\mathbf{u})} &= \sum_{j\in\{0,1\}}\Big(\frac{-f(\mathbf{x},\mathbf{u})}{|-f(\mathbf{x},\mathbf{u})|}\Big)^j e_j^{-f(\mathbf{x},\mathbf{u})}\\ &= \sum_{j\in\{0,1\}} i(\mathbf{x},\mathbf{u})^j e_j^{-f(\mathbf{x},\mathbf{u})}.\end{aligned} \tag{13}$$

Definition 6 (Trigonometric transform) *Let $\mathbf{A}:\mathbb{R}^m\to C\ell_{p,q}$ be a multivector field, $\mathbf{x},\mathbf{u}\in\mathbb{R}^m$ vectors, F a finite set of ν, mappings $\mathbb{R}^m\times\mathbb{R}^m\to C\ell_{p,q}$, and $\mathbf{j}\in\{0,1\}^\nu$ a multi-index*[‡]*. The* ***Trigonometric transform*** *(TT) $\mathscr{F}_F$ is defined by*

$$\mathscr{F}_F(\mathbf{A})(\mathbf{u}):=\int_{\mathbb{R}^m}\mathbf{A}(\mathbf{x})\prod_{l=1}^{\nu}e_{j_l}^{-f_l(\mathbf{x},\mathbf{u})}\,\mathrm{d}^m\mathbf{x} \tag{14}$$

with $e_{j_l}^{-f_l(\mathbf{x},\mathbf{u})}\in\mathbb{R}$ from Notation 3.

[†]Note, that in the current paper we generally do not use i as an index, but rather as a function $i(\mathbf{x},\mathbf{u})$, or $i(\mathbf{u})$, or simply a constant i, all squaring to -1, reminiscent of the complex imaginary unit in $\mathbb{C}$.
[‡]Note, that for distinction an index is written in normal font $j,k,\ldots$, and a multi-index is written with bold font $\mathbf{j},\mathbf{k},\ldots$.

Remark 7 *The $e_{j_l}^{-f(\mathbf{x},\mathbf{u})} \in \mathbb{R}$ are in the center of the geometric algebra. Therefore, there is no need to distinguish the order of their appearances. But it may be helpful though in order to stress their relation to the GFT. Let $\mathscr{F}_{F_1^{\mathbf{j}},F_2^{\mathbf{k}}}$ be a geometric Fourier transform and $\mathbf{j} \in \{0,1\}^{\mu}$,$\mathbf{k} \in \{0,1\}^{(\nu-\mu)}$ multi-indices. Then,*

$$\mathscr{F}_{F_1^{\mathbf{j}},F_2^{\mathbf{k}}}(\mathbf{A})(\mathbf{u}) := \int_{\mathbb{R}^m} \prod_{l=1}^{\mu} e_{j_l}^{-f_l(\mathbf{x},\mathbf{u})} \mathbf{A}(\mathbf{x}) \prod_{l=\mu+1}^{\nu} e_{k_l}^{-f_l(\mathbf{x},\mathbf{u})} \, \mathrm{d}^m\mathbf{x} \tag{15}$$

is a trigonometric transform.

Notation 8 *The standard sine transform*

$$\mathscr{F}_s(\mathbf{A})(\mathbf{u}) = \int_{\mathbb{R}^m} \sin(-2\pi\mathbf{x}\cdot\mathbf{u})\mathbf{A}(\mathbf{x})\,\mathrm{d}^m\mathbf{x} = \mathscr{F}_{(2\pi i\mathbf{x}\cdot\mathbf{u})^1}(\mathbf{A})(\mathbf{u}), \tag{16}$$

the standard cosine transform

$$\mathscr{F}_c(\mathbf{A})(\mathbf{u}) = \int_{\mathbb{R}} \cos(-2\pi\mathbf{x}\cdot\mathbf{u})\mathbf{A}(\mathbf{x})\,\mathrm{d}^m\mathbf{x} = \mathscr{F}_{(2\pi i\mathbf{x}\cdot\mathbf{u})^0}(\mathbf{A})(\mathbf{u}), \tag{17}$$

and their compositions in higher dimensions are special cases of the trigonometric transform. For them, we will sometimes write the letters s and c as lower index of the transform for the sake of brevity and to stress their special shape. As an example, we would write $\mathscr{F}_{sc}$ instead of $\mathscr{F}_{(i_1x_1u_1)^1,(i_2x_2u_2)^0}$, which means

$$\mathscr{F}_{sc}(\mathbf{A})(\mathbf{u}) = \int_{\mathbb{R}^2} \sin(-x_1u_1)\cos(-x_2u_2)\mathbf{A}(\mathbf{x})\,\mathrm{d}\mathbf{x}. \tag{18}$$

3. The True Nature of Separable GFT

The definition of separability has already been introduced in [6].

Definition 9 *We call a mapping $f : \mathbb{R}^m \times \mathbb{R}^m \to C\ell_{p,q}$ x-**separable** or separable with respect to its first argument, if it suffices*

$$f = |f(\mathbf{x},\mathbf{u})| i(\mathbf{u}), \tag{19}$$

*where $i : \mathbb{R}^m \to C\ell_{p,q}$ is a function that does not depend on $\mathbf{x}$ and $|f(\mathbf{x},\mathbf{u})| : \mathbb{R}^2 \to \mathbb{R}$ a real valued function as in Notation 3. Analogously we call it **separable** or separable with respect to both arguments, if it suffices*

$$f = |f(\mathbf{x},\mathbf{u})| i, \tag{20}$$

with constant $i \in C\ell_{p,q}, i^2 = -1$.

Analogously, a geometric Fourier transform that consists of separable mappings F_1, F_2 is called separable. Separability is a central quality for multiplication, shift and convolution properties of GFTs. Almost every transform from approach B and C is separable.

Example 10 *Every transform of our standard examples from Example 2, except for the cylindrical Fourier transform for dimensions higher than two, is separable.*

In this section, we want to take a closer look at this vast class of GFTs. By expressing them by means of the trigonometric transforms, we will be able to reveal their true nature: they are combinations of simple real-valued transforms.

Theorem 11 (GFT decomposition into TT) *Any geometric Fourier transform $\mathscr{F}_{F_1,F_2}$ with x-separable mappings $\forall l = 1,\ldots,\nu : f_l(\mathbf{x},\mathbf{u}) = |f_l(\mathbf{x},\mathbf{u})| i_l(\mathbf{u})$ of a multivector field $\mathbf{A}(\mathbf{x}) = \sum_{\mathbf{r}} a_{\mathbf{r}}(\mathbf{x})\mathbf{e}_{\mathbf{r}}$ is the sum of real valued trigonometric transforms $\mathscr{F}_{F_1^{\mathbf{j}},F_2^{\mathbf{k}}}(a_{\mathbf{r}})(\mathbf{u}) \in \mathbb{R}$ with multivector factors*

$$\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) = \sum_{\mathbf{r}} \sum_{\substack{\mathbf{j}\in\{0,1\}^{\mu},\\ \mathbf{k}\in\{0,1\}^{(\nu-\mu)}}} \mathscr{F}_{F_1^{\mathbf{j}},F_2^{\mathbf{k}}}(a_{\mathbf{r}})(\mathbf{u}) \prod_{l=1}^{\mu} i_l(\mathbf{u})^{j_l} \mathbf{e}_{\mathbf{r}} \prod_{l=\mu+1}^{\nu} i_l(\mathbf{u})^{k_l}. \tag{21}$$

Proof: Using Notation 3, we can write any GFT as

$$\begin{aligned}\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) &= \int_{\mathbb{R}^m} \prod_{l=1}^{\mu} e^{-f_l(\mathbf{x},\mathbf{u})} \mathbf{A}(\mathbf{x}) \prod_{l=\mu+1}^{\nu} e^{-f_l(\mathbf{x},\mathbf{u})} \,\mathrm{d}^m\mathbf{x} \\ &\overset{\text{Lem. 4}}{=} \int_{\mathbb{R}^m} \prod_{l=1}^{\mu} \sum_{j_l\in\{0,1\}} \left(\frac{f_l(\mathbf{x},\mathbf{u})}{|f_l(\mathbf{x},\mathbf{u})|}\right)^{j_l} e_{j_l}^{-f_l(\mathbf{x},\mathbf{u})} \mathbf{A}(\mathbf{x}) \prod_{l=\mu+1}^{\nu} \sum_{k_l\in\{0,1\}} \left(\frac{f_l(\mathbf{x},\mathbf{u})}{|f_l(\mathbf{x},\mathbf{u})|}\right)^{k_l} e_{k_l}^{-f_l(\mathbf{x},\mathbf{u})} \,\mathrm{d}^m\mathbf{x}.\end{aligned} \tag{22}$$

Since the GFT has x-separable mappings, we can replace $\frac{f_l(\mathbf{x},\mathbf{u})}{|f_l(\mathbf{x},\mathbf{u})|}$ by $i_l(\mathbf{u})$ with $i_l(\mathbf{u})^2 = -1$ and write

$$\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) = \int_{\mathbb{R}^m} \prod_{l=1}^{\mu} \sum_{j_l\in\{0,1\}} i_l(\mathbf{u})^{j_l} e_{j_l}^{-f(\mathbf{x},\mathbf{u})} \mathbf{A}(\mathbf{x}) \prod_{l=\mu+1}^{\nu} \sum_{k_l\in\{0,1\}} i_l(\mathbf{u})^{k_l} e_{k_l}^{-f(\mathbf{x},\mathbf{u})} \,\mathrm{d}^m\mathbf{x}. \tag{23}$$

We collect the indices $j_l, l = 1,...,\mu$ into the multi-index $\mathbf{j} \in \{0,1\}^{\mu}$ and $k_l, l = \mu+1,...,\nu$ into the multi-index $\mathbf{k} \in \{0,1\}^{\nu-\mu}$ and get

$$\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) = \sum_{\substack{\mathbf{j}\in\{0,1\}^{\mu},\\ \mathbf{k}\in\{0,1\}^{(\nu-\mu)}}} \int_{\mathbb{R}^m} \prod_{l=1}^{\mu} i_l(\mathbf{u})^{j_l} e_{j_l}^{-f_l(\mathbf{x},\mathbf{u})} \mathbf{A}(\mathbf{x}) \prod_{l=\mu+1}^{\nu} e_{j_l}^{-f_l(\mathbf{x},\mathbf{u})} i_l(\mathbf{u})^{k_l} \,\mathrm{d}^m\mathbf{x}. \tag{24}$$

Since the $e_{j_l}^{f_l(\mathbf{x},\mathbf{u})} \in \mathbb{R}$, this leads to

$$\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) = \sum_{\substack{\mathbf{j}\in\{0,1\}^{\mu},\\ \mathbf{k}\in\{0,1\}^{(\nu-\mu)}}} \prod_{l=1}^{\mu} i_l(\mathbf{u})^{j_l} \int_{\mathbb{R}^m} \prod_{l=1}^{\mu} e_{j_l}^{-f_l(\mathbf{x},\mathbf{u})} \mathbf{A}(\mathbf{x}) \prod_{l=\mu+1}^{\nu} e_{j_l}^{-f_l(\mathbf{x},\mathbf{u})} \,\mathrm{d}^m\mathbf{x} \prod_{l=\mu+1}^{\nu} i_l(\mathbf{u})^{k_l} \tag{25}$$

and together with Definition 6 to

$$\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) = \sum_{\substack{\mathbf{j}\in\{0,1\}^{\mu},\\ \mathbf{k}\in\{0,1\}^{(\nu-\mu)}}} \prod_{l=1}^{\mu} i_l(\mathbf{u})^{j_l} \mathscr{F}_{F_1^{\mathbf{j}},F_2^{\mathbf{k}}}(\mathbf{A})(\mathbf{u}) \prod_{l=\mu+1}^{\nu} i_l(\mathbf{u})^{k_l}. \tag{26}$$

The trigonometric transform itself does not contain any multivector and therefore preserves the basis blades of the multivector field $\mathbf{A} = \sum_{\mathbf{r}} a_{\mathbf{r}}\mathbf{e}_{\mathbf{r}}$, i.e.

$$\begin{aligned}\mathscr{F}_{F_1^{\mathbf{j}},F_2^{\mathbf{k}}}(\mathbf{A})(\mathbf{u}) &= \mathscr{F}_{F_1^{\mathbf{j}},F_2^{\mathbf{k}}} \sum_{\mathbf{r}} a_{\mathbf{r}}\mathbf{e}_{\mathbf{r}}(\mathbf{u}) \\ &= \sum_{\mathbf{r}} \mathscr{F}_{F_1^{\mathbf{j}},F_2^{\mathbf{k}}}(a_{\mathbf{r}})(\mathbf{u})\mathbf{e}_{\mathbf{r}}\end{aligned} \tag{27}$$

with $\mathscr{F}_{F_1^{\mathbf{j}},F_2^{\mathbf{k}}}(a_{\mathbf{r}})(\mathbf{u}) \in \mathbb{R}$. As a result the x-separable GFT can be written as

$$\begin{aligned}\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) &= \sum_{\substack{\mathbf{j}\in\{0,1\}^{\mu},\\ \mathbf{k}\in\{0,1\}^{(\nu-\mu)}}} \prod_{l=1}^{\mu} i_l(\mathbf{u})^{j_l} \sum_{\mathbf{r}} \mathscr{F}_{F_1^{\mathbf{j}},F_2^{\mathbf{k}}}(a_{\mathbf{r}})(\mathbf{u})\mathbf{e}_{\mathbf{r}} \prod_{l=\mu+1}^{\nu} i_l(\mathbf{u})^{k_l} \\ &= \sum_{\substack{\mathbf{j}\in\{0,1\}^{\mu},\\ \mathbf{k}\in\{0,1\}^{(\nu-\mu)}}} \sum_{\mathbf{r}} \mathscr{F}_{F_1^{\mathbf{j}},F_2^{\mathbf{k}}}(a_{\mathbf{r}})(\mathbf{u}) \prod_{l=1}^{\mu} i_l(\mathbf{u})^{j_l} \mathbf{e}_{\mathbf{r}} \prod_{l=\mu+1}^{\nu} i_l(\mathbf{u})^{k_l},\end{aligned} \tag{28}$$

which leads to the assertion. □

Remark 12 *The formula (21) can be written more beautifully as*

$$\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) = \sum_{\substack{\mathbf{j}\in\{0,1\}^{\mu},\\ \mathbf{k}\in\{0,1\}^{(\nu-\mu)}}} \prod_{l=1}^{\mu} i_l(\mathbf{u})^{j_l} \mathscr{F}_{F_1^{\mathbf{j}},F_2^{\mathbf{k}}}(\mathbf{A})(\mathbf{u}) \prod_{l=\mu+1}^{\nu} i_l(\mathbf{u})^{k_l}, \tag{29}$$

but we prefere the first version, because it stresses how the transform is done completely in the real numbers.

Corollary 13 *A geometric Fourier transform $\mathscr{F}_{F_1,F_2}$ with separable mappings $\forall l = 1,...,\nu : f_l(\mathbf{x},\mathbf{u}) = i_l|f_l(\mathbf{x},\mathbf{u})|, i_l^2 \in \mathbb{R}^-$ of a multivector field $\mathbf{A}(\mathbf{x}) = \sum_{\mathbf{r}} a_{\mathbf{r}}(\mathbf{x})\mathbf{e}_{\mathbf{r}}$ is the sum of real valued trigonometric transforms $\mathscr{F}_{F_1^{\mathbf{j}},F_2^{\mathbf{k}}}(a_{\mathbf{r}})(\mathbf{u}) \in \mathbb{R}$ with constant multivector factors*

$$\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) = \sum_{\mathbf{r}} \sum_{\substack{\mathbf{j}\in\{0,1\}^{\mu},\\ \mathbf{k}\in\{0,1\}^{(\nu-\mu)}}} \mathscr{F}_{F_1^{\mathbf{j}},F_2^{\mathbf{k}}}(a_{\mathbf{r}})(\mathbf{u}) \prod_{l=1}^{\nu} i_l^{j_l} \mathbf{e}_{\mathbf{r}} \prod_{l=\mu+1}^{\nu} i_l^{k_l}. \tag{30}$$

That means, if we interpret a multivector valued signal as many signals, saved in 2^n channels, the separable geometric Fourier transforms can be interpreted as real valued transforms, that work one after another on each of the channels, get added in a certain way and written into certain channels depending on the multivector factor. As a result they can be interpreted, analyzed, and implemented with the same tools as the classical real-valued transforms.

This corollary shows that the separable geometric Fourier transforms may look difficult, but really are structures hardly more complicated than the classical Fourier transform. They are linear combinations of the real valued trigonometric transforms in a Clifford algebra.

Example 14 *The restrictions of Corollary 13 are fulfilled by all transforms from Example 2, except for the cylindrical transform for dimensions higher than two. All transforms from Example 2, including the cylindrical transform for dimension two, take the following shapes.*

1. *For the Clifford Fourier transform from [9, 10, 11] for multivector fields $A : \mathbb{R}^n \to C\ell_{n,0}$, $n = 2 \pmod 4$ or $n = 3 \pmod 4$ the Corollary takes the form*

$$\begin{aligned}\mathscr{F}_{f_1}(\mathbf{A}) &= \sum_{\mathbf{r}} \sum_{k\in\{0,1\}} \mathscr{F}_{f_1^k}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} i_n^k \\ &= \sum_{\mathbf{r}} \mathscr{F}_c(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} + \mathscr{F}_s(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} i_n,\end{aligned} \tag{31}$$

with n-dimensional sine and cosine transforms.

2. *The transform by Sommen [12, 13] satisfies*

$$\mathscr{F}_{f_1,\dots,f_n}(\mathbf{A}) = \sum_{\mathbf{r}} \sum_{\mathbf{k}\in\{0,1\}^n} \mathscr{F}_{f_1^{k_1},\dots,f_n^{k_n}}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} \prod_{l=1}^{n} \mathbf{e}_l^{k_l}. \tag{32}$$

3. *The quaternionic Fourier transform [14] takes the shape*

$$\begin{aligned}\mathscr{F}_{f_1,f_2}(\mathbf{A}) &= \sum_{\mathbf{r}} \sum_{j,k\in\{0,1\}} \boldsymbol{i}^j \mathscr{F}_{f_1^j,f_2^k}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} \boldsymbol{j}^k \\ &= \sum_{\mathbf{r}} \mathscr{F}_{cc}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} + \mathscr{F}_{cs}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}}\boldsymbol{j} + \boldsymbol{i}\mathscr{F}_{sc}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} + \boldsymbol{i}\mathscr{F}_{ss}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}}\boldsymbol{j}.\end{aligned} \tag{33}$$

4. *Analogously, the spacetime Fourier transform [16] takes the shape*

$$\begin{aligned}\mathscr{F}_{f_1,f_2}(\mathbf{A}) &= \sum_{\mathbf{r}} \sum_{j,k\in\{0,1\}} \mathbf{e}_4^j \mathscr{F}_{f_1^j,f_2^k}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}}(\varepsilon_4\mathbf{e}_4 i_4)^k \\ &= \sum_{\mathbf{r}} \mathscr{F}_{cc}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} + \mathscr{F}_{cs}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}}\varepsilon_4\mathbf{e}_4 i_4 + \mathbf{e}_4\mathscr{F}_{sc}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} + \mathbf{e}_4\mathscr{F}_{ss}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}}\varepsilon_4\mathbf{e}_4 i_4.\end{aligned} \tag{34}$$

5. *The Clifford Fourier transform for color images [7] with bivector B takes the form*

$$\begin{aligned}\mathscr{F}_{f_1,f_2,f_3,f_4}(\mathbf{A}) &= \sum_{\mathbf{r}} \sum_{\mathbf{j},\mathbf{k}\in\{0,1\}^2} B^{j_1}(iB)^{j_2} \mathscr{F}_{f_1^{j_1},f_2^{j_2},f_3^{k_1},f_4^{j_3}}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} B^{k_1}(iB)^{k_2}. \\ &= \sum_{\mathbf{r}} \mathscr{F}_{cccc}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} + \mathscr{F}_{cccs}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} iB + \mathscr{F}_{ccsc}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} B + \mathscr{F}_{ccss}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} BiB \\ &\quad + iB\mathscr{F}_{cscc}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} + iB\mathscr{F}_{cscs}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} iB + iB\mathscr{F}_{cssc}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} B + iB\mathscr{F}_{csss}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} BiB \\ &\quad + B\mathscr{F}_{sccc}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} + B\mathscr{F}_{sccs}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} iB + B\mathscr{F}_{scsc}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} B + B\mathscr{F}_{scss}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} BiB \\ &\quad + BiB\mathscr{F}_{sscc}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} + BiB\mathscr{F}_{sscs}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} iB + BiB\mathscr{F}_{sssc}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} B + BiB\mathscr{F}_{ssss}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} BiB.\end{aligned} \tag{35}$$

Please note that here expressions like $\mathscr{F}_{cccs},\dots$ refer to two-dimensional sine and cosine transforms with respect to the variables $x_1u_1 + x_2u_2$ each.

6. *The cylindrical Fourier transform [17] is not separable except for the case $n = 2$. Here, it satisfies*

$$\begin{aligned}\mathscr{F}_{f_1}(\mathbf{A}) &= \sum_{\mathbf{r}} \sum_{j\in\{0,1\}} (\mathbf{e}_{12})^{j_l} \mathscr{F}_{f_1^j}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} \\ &= \sum_{\mathbf{r}} \mathscr{F}_c(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} + \mathbf{e}_{12}\mathscr{F}_s(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}},\end{aligned} \tag{36}$$

with sine and cosine transforms of the variables $|\mathbf{x}\wedge\mathbf{u}|$. But for all other dimensions, no closed formula can be constructed in a similar way.

4. The explicitly invertible GFT

There is another important class of GFTs, which we want to take a closer look at.

Definition 15 *Let $F_1 := \{i_1, ..., i_\mu\}, F_2 := \{i_{\mu+1}, ..., i_m\}$ be two ordered finite sets of square roots of minus one, $i_k \in C\ell_{p,q}, \forall k = 1, ..., m : i_k^2 = -1$, then we denote the transform*

$$\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) := (2\pi)^{-\frac{m}{2}} \int_{\mathbb{R}^m} \left(\prod_{k=1}^{\mu} e^{-i_k x_k u_k} \right) \mathbf{A}(\mathbf{x}) \left(\prod_{k=\mu+1}^{m} e^{-i_k x_k u_k} \right) \mathrm{d}^m \mathbf{x} \tag{37}$$

*of a function $f : \mathbb{R}^m \to C\ell_{p,q}$ by **explicitly invertible geometric Fourier transform** (EIGFT).*

This definition is a special case of the general geometric Fourier transform from Definition 1 and of the separable GFT from the previous section. Please note that it also covers many transforms that can be brought into this form by renaming the components of $\mathbf{x}$ and $\mathbf{u}$ or by substitution of the integration variables. For example the factor (2π) can be removed from the front into the exponentials by substitution. We chose to use this shape because it has played a special role in [21] and [2], where this shape was used. This restricted version of Definition 1 is so popular because the additional claims guarantee that the inverse transform of any EIGFT is an EIGFT itself, namely

$$\mathscr{F}_{F_1,F_2}^{-1} = \mathscr{F}_{\{-i_\mu,...,-i_1\},\{-i_m,...,-i_{\mu+1}\}}. \tag{38}$$

Even though we do no know yet if the restrictions in Definition 15 are necessary to guarantee the existence of a GFT inverse, so far it is the only restrictions that we have. Therefore, this class of GFTs is very important because the applications of a transform that puts a function into a space from which it may never return are rather sparse. We were able to construct a bijective transform, that that does not satisfy the Definition 15. But if there exists a bijective GFT whose inverse is a GFT itself that does not satisfy Definition 15 is a matter of current research.

Example 16 *Our standard examples of geometric Fourier transforms form Example 2 satisfy the restricted definition of a EIGFT, except for the Clifford Fourier transform for color images [7] and the cylindrical Fourier transform [17].*

It is easy to show that the color image transform is not bijective if we look at real valued functions $a(\mathbf{x}) : \mathbb{R}^m \to \mathbb{R}$. Since they are in the center of the Clifford algebra, the exponentials cancel each other out

$$\begin{aligned} \mathscr{F}_{f_1,f_2,f_3,f_4}(A)(\mathbf{u}) &= \int_{\mathbb{R}^2} e^{\frac{1}{2}(x_1u_1+x_2u_2)B} e^{\frac{1}{2}(x_1u_1+x_2u_2)iB} a(\mathbf{x}) e^{-\frac{1}{2}(x_1u_1+x_2u_2)iB} e^{-\frac{1}{2}(x_1u_1+x_2u_2)B} \,\mathrm{d}^2\mathbf{x} \\ &= \int_{\mathbb{R}^2} a(\mathbf{x}) e^{\frac{1}{2}(x_1u_1+x_2u_2)B} e^{\frac{1}{2}(x_1u_1+x_2u_2)iB} e^{-\frac{1}{2}(x_1u_1+x_2u_2)iB} e^{-\frac{1}{2}(x_1u_1+x_2u_2)B} \,\mathrm{d}^2\mathbf{x} \\ &= \int_{\mathbb{R}^2} a(\mathbf{x}) \,\mathrm{d}^2\mathbf{x}. \end{aligned} \tag{39}$$

The functions are transformed to one constant real number, their integral. So, this GFT is clearly not bijective. Batard et al. use the transform for vector valued functions only, for which it is invertible.

Whether or not the cylindrical Fourier transform for dimensions higher than two can be inverted is not that easy to decide.

Theorem 17 (EIGFT decomposition into classic TT) *A explicitly invertible geometric Fourier transform of a multivector field is the sum of the composition of standard 1D sine and cosine transforms with constant multivector factors*

$$\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) = \sum_{\mathbf{r}} \sum_{\mathbf{j}\in\{0,1\}^m} \mathscr{F}_{(i_1x_1u_1)^{j_1}} \left(... \mathscr{F}_{(i_mx_mu_m)^{j_m}} (a_{\mathbf{r}}(\mathbf{x}))(\mathbf{u})) \right) \prod_{l=1}^{\mu} i_l^{j_l} \mathbf{e}_{\mathbf{r}} \prod_{l=\mu+1}^{m} i_l^{j_l}. \tag{40}$$

Proof: We start out with Theorem 11 .

$$\begin{aligned} \mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) &= \sum_{\mathbf{r}} \sum_{\substack{\mathbf{j}\in\{0,1\}^\mu,\\ \mathbf{k}\in\{0,1\}^{(\nu-\mu)}}} \mathscr{F}_{F_1^{\mathbf{j}},F_2^{\mathbf{k}}}(a_{\mathbf{r}})(\mathbf{u}) \prod_{l=1}^{\mu} i_l(\mathbf{u})^{j_l} \mathbf{e}_{\mathbf{r}} \prod_{l=\mu+1}^{\nu} i_l(\mathbf{u})^{k_l} \\ &= \sum_{\mathbf{r}} \sum_{\substack{\mathbf{j}\in\{0,1\}^\mu,\\ \mathbf{k}\in\{0,1\}^{(\nu-\mu)}}} \int_{\mathbb{R}^m} \prod_{l=1}^{\mu} e_{j_l}^{-f_l(\mathbf{x},\mathbf{u})} a_{\mathbf{r}}(\mathbf{x}) \prod_{l=\mu+1}^{\nu} e_{k_l}^{-f_l(\mathbf{x},\mathbf{u})} \,\mathrm{d}^m\mathbf{x} \prod_{l=1}^{\mu} i_l(\mathbf{u})^{j_l} \mathbf{e}_{\mathbf{r}} \prod_{l=\mu+1}^{\nu} i_l(\mathbf{u})^{k_l} \end{aligned} \tag{41}$$

and insert the special shape of the EIGFT. Here $\nu = m$ and the factor $(2\pi)^{-\frac{m}{2}}$ appears in front of the integral and not in the exponentials

$$\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) = (2\pi)^{-\frac{m}{2}} \sum_{\mathbf{r}} \sum_{\substack{\mathbf{j}\in\{0,1\}^\mu,\\ \mathbf{k}\in\{0,1\}^{(m-\mu)}}} \int_{\mathbb{R}^m} \prod_{l=1}^{\mu} e_{j_l}^{-f_l(\mathbf{x},\mathbf{u})} a_{\mathbf{r}}(\mathbf{x}) \prod_{l=\mu+1}^{m} e_{k_l}^{-f_l(\mathbf{x},\mathbf{u})} \,\mathrm{d}^m\mathbf{x} \prod_{l=1}^{\mu} i_l^{j_l} \mathbf{e}_{\mathbf{r}} \prod_{l=\mu+1}^{m} i_l^{k_l}. \tag{42}$$

Comprehension of the multi-indices $\mathbf{j}$ and $\mathbf{k}$ into just one multi- index $\mathbf{j}$ and integration over each of the coordinates of $\mathbf{x}$ separately reveals a composition of one-dimensional transforms

$$\begin{aligned}\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) =&(2\pi)^{-\frac{m}{2}}\sum_{\mathbf{r}}\sum_{\mathbf{j}\in\{0,1\}^m}\int_{\mathbb{R}^m}\prod_{l=1}^{m} e_{j_l}^{-f_l(\mathbf{x},\mathbf{u})} a_{\mathbf{r}}(\mathbf{x})\,\mathrm{d}^m\mathbf{x}\prod_{l=1}^{\mu} i_l^{j_l}\mathbf{e}_{\mathbf{r}}\prod_{l=\mu+1}^{m} i_l^{j_l}\\ =&(2\pi)^{-\frac{m}{2}}\sum_{\mathbf{r}}\sum_{\mathbf{j}\in\{0,1\}^m}\int_{\mathbb{R}} e_{j_1}^{-i_1x_1u_1}\dots\int_{\mathbb{R}} e_{j_m}^{-i_mx_mu_m} a_{\mathbf{r}}(\mathbf{x})\,\mathrm{d}x_m\dots\mathrm{d}x_1\prod_{l=1}^{\mu} i_l^{j_l}\mathbf{e}_{\mathbf{r}}\prod_{l=\mu+1}^{m} i_l^{j_l}\\ =&\sum_{\mathbf{r}}\sum_{\mathbf{j}\in\{0,1\}^m}\mathscr{F}_{(i_1x_1u_1)^{j_1}}\left(\dots\mathscr{F}_{(i_mx_mu_m)^{j_m}}(a_{\mathbf{r}}(\mathbf{x}))(\mathbf{u})\right)\Big)\prod_{l=1}^{\mu} i_l^{j_l}\mathbf{e}_{\mathbf{r}}\prod_{l=\mu+1}^{m} i_l^{j_l}.\end{aligned} \tag{43}$$

Depending on whether the multi-indices take the value one or zero, each appearing transform is now a classical one-dimensional sine transform $\mathscr{F}_s$ or cosine transform $\mathscr{F}_c$ of a real valued function. ☐ This decomposition directly shows how a classical FFT algorithm can be applied to efficiently calculate each of the EIGFTs.

Example 18 *Theorem 17 can be applied to most of the standard examples from Example 2.*

1. *The decomposition into the 1D sine and cosine transforms for the Clifford Fourier transform from [9, 10, 11] can be written as*

$$\begin{aligned}\mathscr{F}_{f_1}(\mathbf{A}) &= \int_{\mathbb{R}^n}\mathbf{A}(\mathbf{x})e^{2\pi i_n\mathbf{x}\cdot\mathbf{u}}\,\mathrm{d}^n\mathbf{x}\\ &=\int_{\mathbb{R}^n}\mathbf{A}(\mathbf{x})\prod_{j=1}^{n} e^{2\pi i_n x_j u_j}\,\mathrm{d}^n\mathbf{x}\\ &=\sum_{\mathbf{r}}\sum_{\mathbf{j}\in\{0,1\}^n}\mathscr{F}_{(i_nx_1u_1)^{j_1}}\dots\mathscr{F}_{(i_nx_nu_n)^{j_n}}(a_{\mathbf{r}})\mathbf{e}_{\mathbf{r}} i_n^{|\mathbf{j}|}.\end{aligned} \tag{44}$$

   But obviously, the equivalent decomposition into higher dimensional sine and cosine transforms like in Example 14 is much more beautiful.
2. *The formula for the the Sommen Fourier transform was already shown in Example 14.*
3. *The formula for the quaternionic transform was already shown in Example 14.*
4. *The formula for the spacetime Fourier transform was already shown in Example 14.*
5. *The color image transform does not suffice the constraints of Theorem 17. It can only be decomposed as shown in Example 14.*
6. *The shape for the cylindrical transform with dimension $n = 2$ can be given using a little trick.*

$$\begin{aligned}\mathscr{F}_{f_1}(\mathbf{A})(\mathbf{u}) &= \int_{\mathbb{R}^2} A(\mathbf{x})e^{\mathbf{x}\wedge\mathbf{u}}\,\mathrm{d}^2\mathbf{x}\\ &=\int_{\mathbb{R}}\int_{\mathbb{R}} A(x_1,x_2)e^{\mathbf{e}_{12}(x_1u_2-x_2u_1)}\,\mathrm{d}x_1\,\mathrm{d}x_2\end{aligned} \tag{45}$$

   We define the vector $\mathbf{u}^* = \mathbf{u}\mathbf{e}_{12}^{-1} = \mathbf{u}(-\mathbf{e}_{12}) = \mathbf{e}_{12}\mathbf{u} = u_2\mathbf{e}_1 - u_1\mathbf{e}_2$, which satisfies $x\cdot\mathbf{u}^* = x_1u_2 - x_2u_1$ and get

$$\begin{aligned}\mathscr{F}_{f_1}(\mathbf{A})(\mathbf{u}) &= \int_{\mathbb{R}}\int_{\mathbb{R}} A(x_1,x_2)e^{\mathbf{e}_{12}x\cdot\mathbf{u}^*}\,\mathrm{d}x_1\,\mathrm{d}x_2\\ &=\sum_{\mathbf{r}}\mathscr{F}_c(a_{\mathbf{r}})(\mathbf{u}^*)\mathbf{e}_{\mathbf{r}}+\mathscr{F}_s(a_{\mathbf{r}})(\mathbf{u}^*)\mathbf{e}_{\mathbf{r}}\mathbf{e}_{12}\\ &=\sum_{\mathbf{r}}\mathscr{F}_c(a_{\mathbf{r}})(\mathbf{e}_{12}\mathbf{u})\mathbf{e}_{\mathbf{r}}+\mathscr{F}_s(a_{\mathbf{r}})(\mathbf{e}_{12}\mathbf{u})\mathbf{e}_{\mathbf{r}}\mathbf{e}_{12}.\end{aligned} \tag{46}$$

5. Convolution theorem for not coorthogonal exponents

So far in this paper, the description of the GFTs by means of the TT has mainly lead to the negative meaning of demystification. We showed, that most of the GFTs do not differ very much from the real valued trigonometric transforms. That is why, one may regard them as not very interesting. Now, it is time to use the positive side of the demystification and exploit the simplicity to derive new properties, that could not be found without it.

In [18], we presented a convolution theorem for general geometric Fourier transforms with F_1, F_2 being coorthogonal, separable and linear with respect to the first argument. Coorthogonality can be interpreted as mutual commutation or anticommutation among the functions.

Example 19 *Every GFT in Example 2 is coorthogonal.*

Although coorthogonality is fulfilled by almost every popular geometric Fourier transform, we want to deduce a convolution theorem that holds for functions that are separable and linear with respect to the first argument but have arbitrary commutation properties. This formulation of the theorem is especially useful for the treatment of steerable Fourier transforms over the manifolds of square roots of minus one as in [21, 22], which are generally not coorthogonal.

Definition 20 (convolution) *Let $\mathbf{A}(\mathbf{x}),\mathbf{B}(\mathbf{x}):\mathbb{R}^m\to C\ell_{p,q}$ be two multivector fields. Their* ***convolution*** *$(\mathbf{A}*\mathbf{B})(\mathbf{x})$ is defined as*

$$(\mathbf{A}*\mathbf{B})(\mathbf{x}) := \int_{\mathbb{R}^m} \mathbf{A}(\mathbf{y})\mathbf{B}(\mathbf{x}-\mathbf{y})\,\mathrm{d}^m\mathbf{y}. \tag{47}$$

Theorem 21 (convolution) *Let $\mathbf{A},\mathbf{B},\mathbf{C}:\mathbb{R}^m\to C\ell_{p,q}$ be multivector fields with $\mathbf{A}(\mathbf{x})=(\mathbf{C}*\mathbf{B})(\mathbf{x})$ and F_1,F_2 be separable and linear with respect to the first argument, then the geometric Fourier transform of $\mathbf{A}$ satisfies the convolution property*

$$\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) = \sum_{\substack{\mathbf{j},\mathbf{j'}\in\{0,1\}^{\mu}\\ \mathbf{k},\mathbf{k'}\in\{0,1\}^{(\nu-\mu)}}} \prod_{l=1}^{\mu} i_l(\mathbf{u})^{j_l+j'_l} \mathscr{F}_{F_1^{\mathbf{j}},F_2^{\mathbf{k}}}(\mathbf{C})(\mathbf{u}) \mathscr{F}_{F_1^{\mathbf{j'}},F_2^{\mathbf{k'}}}(\mathbf{B})(\mathbf{u}) \prod_{l=\mu+1}^{\nu} i_l(\mathbf{u})^{k_l+k'_l}. \tag{48}$$

Proof: The transform satisfies

$$\begin{aligned}\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) &= \int_{\mathbb{R}^m} \prod_{f\in F_1} e^{-f(\mathbf{x},\mathbf{u})} (\mathbf{C}*\mathbf{B})(\mathbf{x}) \prod_{f\in F_2} e^{-f(\mathbf{x},\mathbf{u})}\,\mathrm{d}^m\mathbf{x}\\ &= \int_{\mathbb{R}^m} \prod_{f\in F_1} e^{-f(\mathbf{x},\mathbf{u})} \int_{\mathbb{R}^m} \mathbf{C}(\mathbf{y})\mathbf{B}(\mathbf{x}-\mathbf{y})\,\mathrm{d}^m\mathbf{y} \prod_{f\in F_2} e^{-f(\mathbf{x},\mathbf{u})}\,\mathrm{d}^m\mathbf{x}\\ &\overset{\mathbf{x}-\mathbf{y}=\mathbf{z}}{=} \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \prod_{f\in F_1} e^{-f(\mathbf{z}+\mathbf{y},\mathbf{u})}\mathbf{C}(\mathbf{y})\mathbf{B}(\mathbf{z}) \prod_{f\in F_2} e^{-f(\mathbf{z}+\mathbf{y},\mathbf{u})}\,\mathrm{d}^m\mathbf{y}\,\mathrm{d}^m\mathbf{z}.\end{aligned} \tag{49}$$

Since the functions in F_1 and F_2 are all linear and separable with respect to the first argument, (49) is equivalent to

$$\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \prod_{f\in F_1} (e^{-f(\mathbf{z},\mathbf{u})}e^{-f(\mathbf{y},\mathbf{u})})\mathbf{C}(\mathbf{y})\mathbf{B}(\mathbf{z}) \prod_{f\in F_2} (e^{-f(\mathbf{y},\mathbf{u})}e^{-f(\mathbf{z},\mathbf{u})})\,\mathrm{d}^m\mathbf{y}\,\mathrm{d}^m\mathbf{z}. \tag{50}$$

Now, we apply (22) with $\frac{f_l(\mathbf{x},\mathbf{u})}{|f_l(\mathbf{x},\mathbf{u})|} = i_l(\mathbf{u})$ and get

$$\begin{aligned}\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) = \sum_{\substack{\mathbf{j},\mathbf{j'}\in\{0,1\}^{\mu}\\ \mathbf{k},\mathbf{k'}\in\{0,1\}^{(\nu-\mu)}}} &\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \prod_{l=1}^{\mu} i_l(\mathbf{u})^{j'_l} e_{j'_l}^{-f_l(\mathbf{z},\mathbf{u})} i_l(\mathbf{u})^{j_l} e_{j_l}^{-f_l(\mathbf{y},\mathbf{u})} \mathbf{C}(\mathbf{y})\mathbf{B}(\mathbf{z})\\ &\prod_{l=\mu+1}^{\nu} i_l(\mathbf{u})^{k_l} e_{k_l}^{-f_l(\mathbf{y},\mathbf{u})} i_l(\mathbf{u})^{k'_l} e_{k'_l}^{-f_l(\mathbf{z},\mathbf{u})}\,\mathrm{d}^m\mathbf{y}\,\mathrm{d}^m\mathbf{z}.\end{aligned} \tag{51}$$

The separated exponentials are real valued and therefore in the center of the geometric algebra. So, this is equivalent to

$$\begin{aligned}\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) = \sum_{\substack{\mathbf{j},\mathbf{j'}\in\{0,1\}^{\mu}\\ \mathbf{k},\mathbf{k'}\in\{0,1\}^{(\nu-\mu)}}} &\prod_{l=1}^{\mu} i_l(\mathbf{u})^{j_l+j'_l} \int_{\mathbb{R}^m} \prod_{l=1}^{\mu} e_{j_l}^{-f_l(\mathbf{y},\mathbf{u})} \mathbf{C}(\mathbf{y}) \prod_{l=\mu+1}^{\nu} e_{k_l}^{-f_l(\mathbf{y},\mathbf{u})}\,\mathrm{d}^m\mathbf{y}\\ &\int_{\mathbb{R}^m} \prod_{l=1}^{\mu} e_{j'_l}^{-f_l(\mathbf{z},\mathbf{u})} \mathbf{B}(\mathbf{z}) \prod_{l=\mu+1}^{\nu} e_{k'_l}^{-f_l(\mathbf{z},\mathbf{u})}\,\mathrm{d}^m\mathbf{z} \prod_{l=\mu+1}^{\nu} i_l(\mathbf{u})^{k_l+k'_l}.\end{aligned} \tag{52}$$

Using Definition 6, leads to

$$\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) = \sum_{\substack{\mathbf{j},\mathbf{j'}\in\{0,1\}^{\mu}\\ \mathbf{k},\mathbf{k'}\in\{0,1\}^{(\nu-\mu)}}} \prod_{l=1}^{\mu} i_l(\mathbf{u})^{j_l+j'_l} \mathscr{F}_{F_1^{\mathbf{j}},F_2^{\mathbf{k}}}(\mathbf{C})(\mathbf{u}) \mathscr{F}_{F_1^{\mathbf{j'}},F_2^{\mathbf{k'}}}(\mathbf{B})(\mathbf{u}) \prod_{l=\mu+1}^{\nu} i_l(\mathbf{u})^{k_l+k'_l}, \tag{53}$$

which completes the proof. □

Corollary 22 (convolution) *Let $\mathbf{A},\mathbf{B},\mathbf{C}:\mathbb{R}^m\to C\ell_{p,q}$ be multivector fields with $\mathbf{A}(\mathbf{x})=(\mathbf{C}*\mathbf{B})(\mathbf{x})$ and F_1,F_2 each consist of functions in the center of $C\ell_{p,q}$, then the the GFT satisfy the simple product formula*

$$\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) = \mathscr{F}_{F_1,F_2}(\mathbf{C})(\mathbf{u})\mathscr{F}_{F_1,F_2}(\mathbf{B})(\mathbf{u}). \tag{54}$$

Remark 23 *For all $C\ell_{p,q}$, $p+q=n$ even, the center is trivial $\{1\}$. For n odd, the center has two elements $\{1,i_n\}$, with i_n the pseudoscalar. In general, the scalar part of every square root of -1 is zero. So, square roots exist in the center of $C\ell_{p,q}$ only for $p+q=n$ odd and they will be proportional to i_n .*

The convolution theorem Theorem 21 is not simply a generalization of the convolution theorem from [18]. That means in the case of coorthogonal functions F_1, F_2, it will not reduce to the other theorem. Here, the exponentials are decomposed, while the other theorem decomposes the multivector functions.

Example 24 *Theorem 21 can take various shapes, depending on different GFTs. We shorten the formulae and stress the simplicity using Notation 8 for the standard sine and cosine transforms.*

1. *The Clifford Fourier transform from [9, 10, 11] takes the form*

$$\begin{aligned}\mathscr{F}_{f_1}(\mathbf{A}) = &\sum_{j,j'\in\{0,1\}} i^{j+j'} \mathscr{F}_{f_1^j}(\mathbf{C})\mathscr{F}_{f_1^{j'}}(\mathbf{B}) \\ =&\mathscr{F}_c(\mathbf{C})\mathscr{F}_c(\mathbf{B}) - \mathscr{F}_s(\mathbf{C})\mathscr{F}_s(\mathbf{B}) + i(\mathscr{F}_c(\mathbf{C})\mathscr{F}_s(\mathbf{B}) + \mathscr{F}_s(\mathbf{C})\mathscr{F}_c(\mathbf{B}))\end{aligned} \tag{55}$$

   for $n = 2 \pmod 4$. For $n = 3 \pmod 4$ we can apply Corollary 22

$$\mathscr{F}_{f_1}(\mathbf{A}) = \mathscr{F}_{f_1}(\mathbf{C})\mathscr{F}_{f_1}(\mathbf{B}). \tag{56}$$

   Because in this case, the pseudoscalar is in the center of $C\ell_{n,0}$.
2. *The transform by Sommen [12, 13] satisfies*

$$\mathscr{F}_{f_1,\dots,f_n}(\mathbf{A}) = \sum_{\mathbf{k},\mathbf{k}'\in\{0,1\}^n} \mathscr{F}_{f_1^{k_1},\dots,f_n^{k_n}}(\mathbf{C})\mathscr{F}_{f_1^{k_1'},\dots,f_n^{k_n'}}(\mathbf{B}) \prod_{l=1}^{n} \mathbf{e}_l^{k_l+k_l'}. \tag{57}$$

3. *The quaternionic Fourier transform [14, 13] has the shape*

$$\mathscr{F}_{f_1,f_2}(\mathbf{A}) = \sum_{j,j',k,k'\in\{0,1\}} \boldsymbol{i}^{j+j'} \mathscr{F}_{f_1^j,f_2^k}(\mathbf{C})\mathscr{F}_{f_1^{j'},f_2^{k'}}(\mathbf{B})\boldsymbol{j}^{k+k'} \tag{58}$$

   which can be explicitly written as

$$\begin{aligned}\mathscr{F}_{f_1,f_2}(\mathbf{A}) =&\mathscr{F}_{cc}(\mathbf{C})\mathscr{F}_{cc}(\mathbf{B}) - \mathscr{F}_{sc}(\mathbf{C})\mathscr{F}_{sc}(\mathbf{B}) - \mathscr{F}_{cs}(\mathbf{C})\mathscr{F}_{cs}(\mathbf{B}) + \mathscr{F}_{ss}(\mathbf{C})\mathscr{F}_{ss}(\mathbf{B}) \\ &+ i(\mathscr{F}_{sc}(\mathbf{C})\mathscr{F}_{cc}(\mathbf{B}) + \mathscr{F}_{cc}(\mathbf{C})\mathscr{F}_{sc}(\mathbf{B}) - \mathscr{F}_{cs}(\mathbf{C})\mathscr{F}_{ss}(\mathbf{B}) - \mathscr{F}_{ss}(\mathbf{C})\mathscr{F}_{cs}(\mathbf{B})) \\ &+ (\mathscr{F}_{cc}(\mathbf{C})\mathscr{F}_{cs}(\mathbf{B}) + \mathscr{F}_{cs}(\mathbf{C})\mathscr{F}_{cc}(\mathbf{B}) - \mathscr{F}_{sc}(\mathbf{C})\mathscr{F}_{ss}(\mathbf{B}) - \mathscr{F}_{ss}(\mathbf{C})\mathscr{F}_{sc}(\mathbf{B}))j \\ &+ i(\mathscr{F}_{cc}(\mathbf{C})\mathscr{F}_{ss}(\mathbf{B}) + \mathscr{F}_{cs}(\mathbf{C})\mathscr{F}_{sc}(\mathbf{B}) + \mathscr{F}_{sc}(\mathbf{C})\mathscr{F}_{cs}(\mathbf{B}) + \mathscr{F}_{ss}(\mathbf{C})\mathscr{F}_{cc}(\mathbf{B}))j.\end{aligned} \tag{59}$$

4. *The convolution theorem for the spacetime Fourier transform [16] takes the same shape as for the quaternionic transform*

$$\mathscr{F}_{f_1,f_2}(\mathbf{A}) = \sum_{j,j',\mathbf{k},k'\in\{0,1\}} \mathbf{e}_4^{j+j'} \mathscr{F}_{f_1^j,f_2^k}(\mathbf{C})\mathscr{F}_{f_1^{j'},f_2^{k'}}(\mathbf{B})(\varepsilon_4\mathbf{e}_4 i_4)^{k+k'}. \tag{60}$$

5. *The Clifford Fourier transform for color images [7] with bivector B takes the form*

$$\mathscr{F}_{f_1,f_2,f_3,f_4}(\mathbf{A}) = \sum_{\mathbf{j},\mathbf{k}\in\{0,1\}^2} (B)^{j_1+j_1'}(iB)^{j_2+j_2'} \mathscr{F}_{f_1^{j_1},f_2^{j_2},f_3^{k_1},f_4^{k_2}}(\mathbf{C})\mathscr{F}_{f_1^{j_1'},f_2^{j_2'},f_3^{k_1'},f_4^{k_2'}}(\mathbf{B})(-B)^{k_1+k_1'}(-iB)^{k_2+k_2'}, \tag{61}$$

   which we will not write down summand by summand, because it comprises 256 terms.
6. *The cylindrical Fourier transform [17] is not separable except for the case $n = 2$. Here it satisfies*

$$\begin{aligned}\mathscr{F}_{f_1}(\mathbf{A}) = &\sum_{j,j'\in\{0,1\}} \mathbf{e}_{12}^{j+j'} \mathscr{F}_{f_1^j}(\mathbf{C})\mathscr{F}_{f_1^{j'}}(\mathbf{B}) \\ =&\mathscr{F}_{f_1^0}(\mathbf{C})\mathscr{F}_{f_1^0}(\mathbf{B}) - \mathscr{F}_{f_1^1}(\mathbf{C})\mathscr{F}_{f_1^1}(\mathbf{B}) + \mathbf{e}_{12}(\mathscr{F}_{f_1^0}(\mathbf{C})\mathscr{F}_{f_1^1}(\mathbf{B}) + \mathscr{F}_{f_1^1}(\mathbf{C})\mathscr{F}_{f_1^0}(\mathbf{B})),\end{aligned} \tag{62}$$

   but for all other no closed formula can be constructed in a similar way.

6. Short convolution theorem for not coorthogonal exponents

The number of summands in Theorem 21 is 4^ν. It is possible to formulate versions of the theorem that consist of only 2^ν summands. This formula still holds for GFT with mappings of arbitrary commutation properties. The main difference to the previous formula is that the summands are not real valued. The elementary transform terms are half GFTs, half TTs, and multivector valued.

Theorem 25 (convolution, short) *Let* $\mathbf{A},\mathbf{B},\mathbf{C}:\mathbb{R}^m \to C\ell_{p,q}$ *be multivector fields with* $\mathbf{A}(\mathbf{x}) = (\mathbf{C}*\mathbf{B})(\mathbf{x})$ *and* F_1, F_2 *be separable and linear with respect to the first argument, then the geometric Fourier transform of* $\mathbf{A}$ *satisfies the convolution property*

$$\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) = \sum_{\substack{\mathbf{j}\in\{0,1\}^{\mu}\\ \mathbf{k}\in\{0,1\}^{(\nu-\mu)}}} \mathscr{F}_{-i_l(\mathbf{u})\frac{\pi}{2}j_l+f_l(\mathbf{y},\mathbf{u}),F_2^{\mathbf{k}}}(\mathbf{C})(\mathbf{u})\mathscr{F}_{F_1^{\mathbf{j}},-i_l(\mathbf{u})\frac{\pi}{2}k_l+f_l(\mathbf{y},\mathbf{u})}(\mathbf{B})(\mathbf{u}). \tag{63}$$

with the transforms

$$\begin{aligned}
\mathscr{F}_{F_1^{\mathbf{j}},-i_l(\mathbf{u})\frac{\pi}{2}k_l+f_l(\mathbf{y},\mathbf{u})}(\mathbf{B})(\mathbf{u}) &= \int_{\mathbb{R}^m}\prod_{l=1}^{\mu} e_{j_l}^{-f_l(\mathbf{x},\mathbf{u})}\mathbf{B}(\mathbf{x}) \prod_{l=\mu+1}^{\nu} e^{i_l(\mathbf{u})\frac{\pi}{2}k_l - f_l(\mathbf{y},\mathbf{u})}\,\mathrm{d}^m\mathbf{x}\\
&= \int_{\mathbb{R}^m}\prod_{l=1}^{\mu} e_{j_l}^{-f_l(\mathbf{x},\mathbf{u})}\mathbf{B}(\mathbf{x}) \prod_{l=\mu+1}^{\nu} (i_l(\mathbf{u})^{k_l} e^{-f_l(\mathbf{y},\mathbf{u})})\,\mathrm{d}^m\mathbf{x},\\
\mathscr{F}_{-i_l(\mathbf{u})\frac{\pi}{2}j_l+f_l(\mathbf{y},\mathbf{u}),F_2^{\mathbf{k}}}(\mathbf{C})(\mathbf{u}) &= \int_{\mathbb{R}^m}\prod_{l=1}^{\mu} e^{i_l(\mathbf{u})\frac{\pi}{2}j_l - f_l(\mathbf{y},\mathbf{u})}\mathbf{C}(\mathbf{x}) \prod_{l=\mu+1}^{\nu} e_{k_l}^{-f_l(\mathbf{x},\mathbf{u})}\,\mathrm{d}^m\mathbf{x}\\
&= \int_{\mathbb{R}^m}\prod_{l=1}^{\mu} (i_l(\mathbf{u})^{j_l} e^{-f_l(\mathbf{y},\mathbf{u})})\mathbf{C}(\mathbf{x}) \prod_{l=\mu+1}^{\nu} e_{k_l}^{-f_l(\mathbf{x},\mathbf{u})}\,\mathrm{d}^m\mathbf{x}
\end{aligned} \tag{64}$$

for multi-indices $\mathbf{j}\in\{0,1\}^{\mu}$ *or* $\mathbf{k}\in\{0,1\}^{(\nu-\mu)}$ *and* $e_j^{-f(\mathbf{x},\mathbf{u})}$ *from Notation 3.*

Proof: We start with (50) in the proof of Theorem 21

$$\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) = \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\prod_{f\in F_1}(e^{-f(\mathbf{z},\mathbf{u})}e^{-f(\mathbf{y},\mathbf{u})})\mathbf{C}(\mathbf{y})\mathbf{B}(\mathbf{z})\prod_{f\in F_2}(e^{-f(\mathbf{y},\mathbf{u})}e^{-f(\mathbf{z},\mathbf{u})})\,\mathrm{d}^m\mathbf{y}\,\mathrm{d}^m\mathbf{z}, \tag{65}$$

apply Lemma 4 and $\frac{f_l(\mathbf{x},\mathbf{u})}{|f_l(\mathbf{x},\mathbf{u})|} = i_l(\mathbf{u})$ and get

$$\begin{aligned}
\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) = &\sum_{\substack{\mathbf{j}\in\{0,1\}^{\mu}\\ \mathbf{k}\in\{0,1\}^{(\nu-\mu)}}} \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\prod_{l=1}^{\mu}(i_l(\mathbf{u})^{j_l}e_{j_l}^{-f_l(\mathbf{z},\mathbf{u})}e^{-f_l(\mathbf{y},\mathbf{u})})\mathbf{C}(\mathbf{y})\mathbf{B}(\mathbf{z})\\
&\prod_{l=\mu+1}^{\nu}(e_{k_l}^{-f_l(\mathbf{y},\mathbf{u})}i_l(\mathbf{u})^{k_l}e^{-f_l(\mathbf{z},\mathbf{u})})\,\mathrm{d}^m\mathbf{y}\,\mathrm{d}^m\mathbf{z}\\
= &\sum_{\substack{\mathbf{j}\in\{0,1\}^{\mu}\\ \mathbf{k}\in\{0,1\}^{(\nu-\mu)}}} \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\prod_{l=1}^{\mu}e_{j_l}^{-f_l(\mathbf{z},\mathbf{u})}\prod_{l=1}^{\mu}(i_l(\mathbf{u})^{j_l}e^{-f_l(\mathbf{y},\mathbf{u})})\mathbf{C}(\mathbf{y})\mathbf{B}(\mathbf{z})\\
&\prod_{l=\mu+1}^{\nu}e_{k_l}^{-f_l(\mathbf{y},\mathbf{u})}\prod_{l=\mu+1}^{\nu}(i_l(\mathbf{u})^{k_l}e^{-f_l(\mathbf{z},\mathbf{u})})\,\mathrm{d}^m\mathbf{y}\,\mathrm{d}^m\mathbf{z}.
\end{aligned} \tag{66}$$

Now, we use for $j_l\in\{0,1\}$

$$i_l(\mathbf{u})^{j_l} = e^{\frac{\pi}{2}i_l(\mathbf{u})j_l} \Rightarrow i_l(\mathbf{u})^{j_l}e^{-f_l(\mathbf{z},\mathbf{u})} = e^{i_l(\mathbf{u})(\frac{\pi}{2}j_l+|-f_l(\mathbf{z},\mathbf{u})|)} \tag{67}$$

and get

$$\begin{aligned}
\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u}) = &\sum_{\substack{\mathbf{j}\in\{0,1\}^{\mu}\\ \mathbf{k}\in\{0,1\}^{(\nu-\mu)}}} \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\prod_{l=1}^{\mu}e_{j_l}^{-f_l(\mathbf{z},\mathbf{u})}\prod_{l=1}^{\mu}e^{i_l(\mathbf{u})(\frac{\pi}{2}j_l+|-f_l(\mathbf{y},\mathbf{u})|)}\mathbf{C}(\mathbf{y})\mathbf{B}(\mathbf{z})\\
&\prod_{l=\mu+1}^{\nu}e_{k_l}^{-f_l(\mathbf{y},\mathbf{u})}\prod_{l=\mu+1}^{\nu}e^{i_l(\mathbf{u})(\frac{\pi}{2}k_l+|-f_l(\mathbf{z},\mathbf{u})|)}\,\mathrm{d}^m\mathbf{y}\,\mathrm{d}^m\mathbf{z}\\
= &\sum_{\substack{\mathbf{j}\in\{0,1\}^{\mu}\\ \mathbf{k}\in\{0,1\}^{(\nu-\mu)}}} \int_{\mathbb{R}^m}\prod_{l=1}^{\mu}e^{i_l(\mathbf{u})(\frac{\pi}{2}j_l+|-f_l(\mathbf{y},\mathbf{u})|)}\mathbf{C}(\mathbf{y})\prod_{l=\mu+1}^{\nu}e_{k_l}^{-f_l(\mathbf{y},\mathbf{u})}\,\mathrm{d}^m\mathbf{y}\\
&\int_{\mathbb{R}^m}\prod_{l=1}^{\mu}e_{j_l}^{-f_l(\mathbf{z},\mathbf{u})}\mathbf{B}(\mathbf{z})\prod_{l=\mu+1}^{\nu}e^{i_l(\mathbf{u})(\frac{\pi}{2}k_l+|-f_l(\mathbf{z},\mathbf{u})|)}\,\mathrm{d}^m\mathbf{z}\\
= &\sum_{\substack{\mathbf{j}\in\{0,1\}^{\mu}\\ \mathbf{k}\in\{0,1\}^{(\nu-\mu)}}} \mathscr{F}_{-i_l(\mathbf{u})\frac{\pi}{2}j_l+f_l(\mathbf{y},\mathbf{u}),F_2^{\mathbf{k}}}(\mathbf{C})(\mathbf{u})\mathscr{F}_{F_1^{\mathbf{j}},-i_l(\mathbf{u})\frac{\pi}{2}k_l+f_l(\mathbf{y},\mathbf{u})}(\mathbf{B})(\mathbf{u}),
\end{aligned} \tag{68}$$

which completes the proof. □

Corollary 26 *Let $\mathbf{A},\mathbf{B},\mathbf{C}:\mathbb{R}^m\to C\ell_{p,q}$ be multivector fields with $\mathbf{A}(\mathbf{x})=(\mathbf{C}*\mathbf{B})(\mathbf{x})$ and F_1,F_2 be separable, linear with respect to the first argument and mutually commutative, then the geometric Fourier transform of $\mathbf{A}$ satisfies the convolution property*

$$\mathscr{F}_{F_1,F_2}(\mathbf{A})(\mathbf{u})=\sum_{\substack{\mathbf{j}\in\{0,1\}^\mu\\ \mathbf{k}\in\{0,1\}^{(\nu-\mu)}}}\prod_{l=1}^{\mu}i_l(\mathbf{u})^{j_l}\mathscr{F}_{F_1,F_2^{\mathbf{k}}}(\mathbf{C})(\mathbf{u})\mathscr{F}_{F_1^{\mathbf{j}},F_2}(\mathbf{B})(\mathbf{u})\prod_{l=\mu+1}^{\nu}(i_l(\mathbf{u})^{k_l}. \tag{69}$$

with $e_j^{-f(\mathbf{x},\mathbf{u})}$ from Notation 3.

Example 27 *Theorem 25 can take various shapes, depending on the different GFTs. We show this for the same examples as before.*

1. *The Clifford Fourier transform from [9, 10, 11] takes the form*

$$\mathscr{F}_{f_1}(\mathbf{A})=\mathscr{F}_c(\mathbf{C})\mathscr{F}_{f_1}(\mathbf{B})+\mathscr{F}_s(\mathbf{C})\mathscr{F}_{f_1}(\mathbf{B})i \tag{70}$$

for $n=2\pmod 4$. For $n=3\pmod 4$ we can apply Corollary 22

$$\mathscr{F}_{f_1}(\mathbf{A})=\mathscr{F}_{f_1}(\mathbf{C})\mathscr{F}_{f_1}(\mathbf{B}). \tag{71}$$

Because in this case, the pseudoscalar is in the center of $C\ell_{n,0}$.

2. *The transform by Sommen [12, 13] is the only one of our examples that fulfills all constraints for Theorem 25 but not for Corollary 26. It satisfies*

$$\mathscr{F}_{f_1,\dots,f_n}(\mathbf{A})=\sum_{\mathbf{k}\in\{0,1\}^n}\mathscr{F}_{F_2^{\mathbf{k}}}(\mathbf{C})\mathscr{F}_{-i_1\frac{\pi}{2}k_1+f_1,\dots,-i_n\frac{\pi}{2}k_n+f_n}(\mathbf{B}). \tag{72}$$

3. *The quaternionic Fourier transform [14, 13] has the shape*

$$\begin{aligned}\mathscr{F}_{f_1,f_2}(\mathbf{A})=&\sum_{j,k\in\{0,1\}}i_1^j\mathscr{F}_{f_1,f_2^k}(\mathbf{C})\mathscr{F}_{f_1^j,f_2}(\mathbf{B})i_2^k\\ =&\mathscr{F}_{f_1,c}(\mathbf{C})\mathscr{F}_{c,f_2}(\mathbf{B})+\mathscr{F}_{f_1,s}(\mathbf{C})\mathscr{F}_{c,f_2}(\mathbf{B})j\\ &+i\mathscr{F}_{f_1,c}(\mathbf{C})\mathscr{F}_{s,f_2}(\mathbf{B})+i\mathscr{F}_{f_1,s}(\mathbf{C})\mathscr{F}_{s,f_2}(\mathbf{B})j.\end{aligned} \tag{73}$$

4. *And the spacetime Fourier transform [16] principally the same shape*

$$\begin{aligned}\mathscr{F}_{f_1,f_2}(\mathbf{A})=&\sum_{j,k\in\{0,1\}}\mathbf{e}_4^j\mathscr{F}_{f_1,f_2^k}(\mathbf{C})\mathscr{F}_{f_1^j,f_2}(\mathbf{B})(\varepsilon_4\mathbf{e}_4i_4)^k\\ =&\mathscr{F}_{f_1,c}(\mathbf{C})\mathscr{F}_{c,f_2}(\mathbf{B})+\mathscr{F}_{f_1,s}(\mathbf{C})\mathscr{F}_{c,f_2}(\mathbf{B})\varepsilon_4\mathbf{e}_4i_4\\ &+\mathbf{e}_4\mathscr{F}_{f_1,c}(\mathbf{C})\mathscr{F}_{s,f_2}(\mathbf{B})+\mathbf{e}_4\mathscr{F}_{f_1,s}(\mathbf{C})\mathscr{F}_{s,f_2}(\mathbf{B})\varepsilon_4\mathbf{e}_4i_4.\end{aligned} \tag{74}$$

5. *The Clifford Fourier transform for color images [7] with bivector B takes the form*

$$\mathscr{F}_{f_1,f_2,f_3,f_4}(\mathbf{A})=\sum_{\mathbf{j},\mathbf{k}\in\{0,1\}^2}(B)^{j_1}(iB)^{j_2}\mathscr{F}_{f_1,f_2,f_3^{k_1},f_4^{k_2}}(\mathbf{C})\mathscr{F}_{f_1^{j_1},f_2^{j_2},f_3,f_4}(\mathbf{B})(-B)^{k_1}(-iB)^{k_2}, \tag{75}$$

or explicitly

$$\begin{aligned}\mathscr{F}_{f_1,f_2,f_3,f_4}(\mathbf{A})=&\mathscr{F}_{f_1,f_2,f_3^0,f_4^0}(\mathbf{C})\mathscr{F}_{f_1^0,f_2^0,f_3,f_4}(\mathbf{B})-\mathscr{F}_{f_1,f_2,f_3^0,f_4^1}(\mathbf{C})\mathscr{F}_{f_1^0,f_2^0,f_3,f_4}(\mathbf{B})iB\\ &-\mathscr{F}_{f_1,f_2,f_3^1f_4^0}(\mathbf{C})\mathscr{F}_{f_1^0,f_2^0,f_3,f_4}(\mathbf{B})B+\mathscr{F}_{f_1,f_2,f_3^1f_4^1}(\mathbf{C})\mathscr{F}_{f_1^0,f_2^0,f_3,f_4}(\mathbf{B})BiB\\ &iB\mathscr{F}_{f_1,f_2,f_3^0,f_4^0}(\mathbf{C})\mathscr{F}_{f_1^0,f_2^1,f_3,f_4}(\mathbf{B})-iB\mathscr{F}_{f_1,f_2,f_3^0,f_4^1}(\mathbf{C})\mathscr{F}_{f_1^0,f_2^1,f_3,f_4}(\mathbf{B})iB\\ &-iB\mathscr{F}_{f_1,f_2,f_3^1,f_4^0}(\mathbf{C})\mathscr{F}_{f_1^0,f_2^1,f_3,f_4}(\mathbf{B})B+iB\mathscr{F}_{f_1,f_2,f_3^1,f_4^1}(\mathbf{C})\mathscr{F}_{f_1^0,f_2^1,f_3,f_4}(\mathbf{B})BiB\\ &B\mathscr{F}_{f_1,f_2,f_3^0,f_4^0}(\mathbf{C})\mathscr{F}_{f_1^1,f_2^0,f_3,f_4}(\mathbf{B})-B\mathscr{F}_{f_1,f_2,f_3^0,f_4^1}(\mathbf{C})\mathscr{F}_{f_1^1,f_2^0,f_3,f_4}(\mathbf{B})iB\\ &-B\mathscr{F}_{f_1,f_2,f_3^1,f_4^0}(\mathbf{C})\mathscr{F}_{f_1^1,f_2^0,f_3,f_4}(\mathbf{B})B+B\mathscr{F}_{f_1,f_2,f_3^1,f_4^1}(\mathbf{C})\mathscr{F}_{f_1^1,f_2^0,f_3,f_4}(\mathbf{B})BiB\\ &+BiB\mathscr{F}_{f_1,f_2,f_3^0,f_4^0}(\mathbf{C})\mathscr{F}_{f_1^1,f_2^1,f_3,f_4}(\mathbf{B})-BiB\mathscr{F}_{f_1,f_2,f_3^0,f_4^1}(\mathbf{C})\mathscr{F}_{f_1^1,f_2^1,f_3,f_4}(\mathbf{B})iB\\ &-BiB\mathscr{F}_{f_1,f_2,f_3^1,f_4^0}(\mathbf{C})\mathscr{F}_{f_1^1,f_2^1,f_3,f_4}(\mathbf{B})B+BiB\mathscr{F}_{f_1,f_2,f_3^1,f_4^1}(\mathbf{C})\mathscr{F}_{f_1^1,f_2^1,f_3,f_4}(\mathbf{B})BiB.\end{aligned} \tag{76}$$

6. *The cylindrical Fourier transform [17] is not separable except for the case $n=2$. Here it satisfies*

$$\mathscr{F}_{f_1}(\mathbf{A})=\mathscr{F}_{f_1^0}(\mathbf{C})\mathscr{F}_{f_1}(\mathbf{B})+\mathbf{e}_{12}\mathscr{F}_{f_1^1}(\mathbf{C})\mathscr{F}_{f_1}(\mathbf{B}), \tag{77}$$

but for all other no closed formula can be constructed in a similar way.

7. Conclusions

In this paper, we have lifted the mystery around the separable geometric Fourier transforms. This has two big consequences.

The decomposition of separable GFT into real valued trigonometric transforms in Theorem 11 takes away their veil, most of their magic and some of their fascination. Even though the definition of the GFT covers more than the separable transforms, these are the most popular ones. When it comes to real applications, even the stronger restriction to EIGFTs is essential. For them, the decomposition into classical one-dimensional, real-valued sine transforms and cosine transforms with constant multivector factors in Theorem 17 displays their ease of use.

This insight has not only the sobering side of demystification in the sense of showing their simplicity, but also the positive side of making them easier to understand, analyze and implement. As examples of the advantages of the new point of view on the GFTs, we present two convolution theorems. Until this equivalence to the trigonometric transforms was shown, the convolution theorems of the GFT were only valid for the subclass of the coorthogonal GFT. Now Theorems 21 and 25 give closed expressions for the convolution of any GFT with mappings that are separable and linear in the first argument, but have arbitrary commutation properties. Another advantage is that the decomposition into one-dimensional sine and cosine transforms directly shows how a classical FFT algorithm can be applied to efficiently calculate the transforms.

For future work, this also leads us into three directions. First, to use the simplicity to explore and simplify already found properties for the separable GFTs and to find new ones using the powerful but easy tool of the TT. Second, and more interesting is to concentrate on the transforms that are not separable or at least not EIGFT and find out if there are bijective transforms among them and how they actually differ from real-valued transforms. Third, the techniques developed in this paper may also help to further elucidate Clifford algebra wavelet type of transforms [23, 24, 25, 26].

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